

# Theoretical Properties of the Entanglement in a Strong Coupling Region

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## Abstract

Entanglement entropy is expected to do a suitable order parameter to classify phase structures at zero temperature. Thus, it is interesting to understand theoretical properties of the entanglement entropy in a strong coupling region. We compute entropy in a non-relativistic model with four fermion interactions and spin imbalance in four dimensional lattice with an infinite fermion mass limit from an exact effective potential to obtain the behavior of the entropy in infinite strong coupling limit. The result is zero in infinite strong coupling and finite lattice spacing. The result supports non-trivial topology needs to be considered in the entanglement entropy. We consider two dimensions to know the lattice artifact, and quantum gravity problems. The entanglement entropy in two dimensional gravity theory is the sum of the classical Shannon entropy and usual expectation values of area term. We also use area law to do the necessary condition in quantum gravity theory to argue translational invariance should be included. Finally, we show universal terms of the entanglement entropy should not be affected by a choice of an entangling surface in two dimensional conformal field theory for one interval and some cases

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of multiple intervals, use a geometric method to extend the discussion to generic multiple intervals, and also discuss the holographic entanglement entropy in two dimensional  $CP^{N-1}$  model.

# 1 Introduction

Quantum gravity theory is expected to combining general relativity and quantum principle. Now we already know quantum gravity needs holographic principle. Holographic principle shows that degrees of freedom in a system are encoded in boundary of the system. In statistical systems, the degrees of freedom is proportional to volume of a system. Thus, it is non-trivial to find holographic principle in a system. In other words, holographic principle should restrict our constructions of quantum gravity. One candidate of weakly coupled quantum gravity theory is string theory. String theory gives a conjecture of a realization of holographic principle from anti-de Sitter/Conformal field theory (*AdS/CFT*) correspondence. The *AdS/CFT* correspondence conjectures that the equivalence between weakly coupled bulk theory and strongly coupled conformal field theory. Therefore, the *AdS/CFT* correspondence also gives implications to strongly coupled quantum field theory.

The entanglement entropy in the Yang-Mills gauge theory is problematic in decomposition problems [1, 2] because the Yang-Mills gauge theory has spatial Wilson loop. If we consider the quantum chromodynamics (QCD) in a strongly coupled region, then the entanglement entropy should vanish because the QCD model is color singlet in the region. Therefore, this possibly implies that decomposition problems are not problematic in a strongly coupled region. The other point of view in the decomposition problems and a strong coupled region is to argue that factorization problems occurs due to ultraviolet scale. The large  $N$  or weakly coupled  $CP^{N-1}$  in a two dimensional lattice do not suffer from factorization problems, and this model in *AdS* should dual to strongly coupled conformal field theory if *AdS/CFT* correspondence is correct. Thus, this implies that strongly coupled conformal field theory may not suffer from decomposition problems [3] also.

Entanglement entropy of topological field theory only depends on topology of manifold. Thus, the entanglement entropy of topological field theories is independent of coupling constant. A theory with local interacting in a strong coupling region on lattice does not have local interacting terms between different sites, then the entanglement entropy should vanish, except for topological contributions. Therefore, the entanglement entropy of topological field theories should be a useful tool to classify phase structure in a strong coupling region at zero temperature on lattice.

The entanglement entropy in the Yang-Mills lattice gauge theory obeys area law from the strong coupling expansion. This result gives an interesting geometric understanding as in holographic entanglement entropy. Indeed, if quantum gravity has translational invariance, Poincaré symmetry, causality and finite entanglement entropy, then the entanglement entropy is determined from non-negative constant and area terms [4]. Therefore, area law in the entanglement entropy is very interesting to give deep insight to quantum gravity.

Our goal of this paper is to study the entanglement entropy in a strongly coupled regime. We first consider a non-relativistic four fermion interaction with spin imbalance in infinite strong coupling and fermion mass limits to find that entropy vanishes, consistent with the lattice  $SU(N)$  Yang-Mills gauge theory from the strong coupling expansion. This result supports our conjecture that the entanglement entropy vanish in strongly coupled regions on lattice should exist for trivial topology if interacting terms between different sites disappear under some limits. To study the entanglement entropy in a strong coupling region, the lattice construction is necessary. Thus, we discuss lattice artifact in entanglement entropy of topological field theories. Then we also discuss the entanglement entropy in quantum gravity theory. We compute the entanglement entropy in two dimensional Einstein-Hilbert action from summing all Riemann surfaces, and obtain expected result as in [4]. Then we also argue translational invariance can rule out volume law of the entanglement entropy at zero temperature in an infinite size system without mass scale. Finally, we discuss the dependence of a choice of an entangling surface of universal terms for the entanglement entropy in two dimensional conformal field theory. In two dimensional conformal field theory, we can also use a geometric method [5], symmetry principle [6] and unique mutual information [7] to show the universal terms of the entanglement entropy for single interval should be independent of a choice of an entangling surface [8], this conclusion can also be extended to some multiple intervals cases, and we also discuss generic multiple intervals from the geometric method. In two dimensional bulk theory or  $CP^{N-1}$  model, this theory is also expected to have holographic duality [9] in large  $N$  limit, and is also a two dimensional conformal field theory. Therefore, a choice of centers in  $AdS_2/CFT_1$  possibly does not affect the result of the holographic entanglement entropy. Then we also use dilute gas approximation to find similar behavior of the entanglement entropy in  $CFT_1$  to the entanglement entropy in two dimensional  $CP^{N-1}$  theory.

We first compute entropy in a non-relativistic model with four fermion interaction and spin imbalance in Sec. 2, and discuss entanglement entropy on two dimensional topological lattice models in Sec. 3. We then discuss entanglement entropy of quantum gravity [10] in Sec. 4. We also discuss dependence of a choice of an entangling surface in conformal field theories [10] in Sec. 5. Finally, we give conclusion in Sec. 6. We review an overlap operator in Appendix. A, review two dimensional Einstein-Hilbert action, finite entropy, and  $CP^{N-1}$  model in Appendix. B, C, and D.

## 2 Non-Relativistic Fermion

We consider a non-relativistic fermion theory with four fermion interaction and spin imbalance on four dimensional lattice to compute the entropy from effective potential exactly when we take infinitely fermion mass limit. Then we find entropy vanishes in an infinitely strong coupling limit, consistent with the strong coupling expansion in the lattice Yang-Mills gauge theory.

### 2.1 Action

The continuum Euclidean action is

$$\int d^4x \left[ \psi^\dagger \left( \partial_\tau - \frac{\nabla^2}{2M} - \mu \right) \psi - \frac{1}{2m^2} (\psi^\dagger \psi)^2 \right], \quad (1)$$

where  $M$  is the fermion mass,  $m^2$  controls strength of interaction, and

$$\psi \equiv \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}, \quad \mu \equiv \begin{pmatrix} \mu_\uparrow & 0 \\ 0 & \mu_\downarrow \end{pmatrix}, \quad (2)$$

and the spin imbalance is controlled from different values of up and down chemical potentials ( $\mu_\uparrow$  and  $\mu_\downarrow$ ). To compute the effective potential exactly, we introduce an auxiliary field  $\phi$  as

$$\int d^4x \left[ \psi^\dagger \left( \partial_\tau - \frac{\nabla^2}{2M} - \mu \right) \psi + \frac{m^2}{2} \phi^2 - \psi^\dagger \phi \psi \right]. \quad (3)$$

The entropy can be found when we take infinite fermion mass limit. Then the lattice action action under the limit is

$$\sum_n \psi_n^\dagger \left( \psi_n - \exp(\mu)(1 + \phi_n) \psi_{n-\hat{e}_0} \right) + \frac{m^2}{2} \phi_n^2. \quad (4)$$

Because we work in the case of finite lattice spacing or finite momentum cut-off, the infinite fermion mass limit is well-defined. Then the partition function [11] is

$$\int D\phi \exp\left(-\frac{m^2}{2} \sum_n \phi_n^2\right) \det\left(\tilde{K}(\mu_\uparrow)^T \tilde{K}(\mu_\downarrow)\right), \quad (5)$$

where

$$(K\psi)_n = \psi_n - \exp(\mu)(1 + \phi_n)\psi_{n-\hat{e}_0}, \quad (6)$$

$\tilde{K}$  only acts on the coordinates space (or it does not act on spinor indices),  $\hat{e}_0$  is the unit vector of the time direction, and we also denote integers by  $i$ - $z$ .

## 2.2 Entropy

The exact solutions on lattice are hard to compute usually, but if we take infinitely heavy fermion mass limit, then the spacial derivative will vanish to simplify our computation. If we have two regions  $A$  and  $B$  on lattice, and want to obtain the entropy in the region  $A$ , then we need to compute the partition function of  $n$ -sheet [12]. The period of the  $A$  region is  $nN_\tau$  and  $B$  region is  $N_\tau$ , where  $N_\tau$  is lattice size of time. Then we also have cuts at time slices  $t = kN_\tau$ , where  $k = 0, 1, \dots, n$  in the region  $B$ . Regions  $A$  and  $B$  share the same boundary. In this set-up, we can find the partition function of region  $A$  region  $B$  can be computed separately because we do not have spacial derivative terms. Then the region  $B$  will not contribute to the entanglement entropy of the region  $A$ . Hence, we only need to compute the partition function in region  $A$ . Because we only consider mean field level or effective potential, we ignore quantum fluctuation of  $\phi_n$ . Then we also have

$$\ln \det A = \text{Tr} \ln A = \sum_p \ln A_p, \quad (7)$$

where  $A$  is a hermitian matrix and  $p$  is momentum. Therefore, we can compute determinant from the effective potential. The effective potential is

$$\begin{aligned} V_{eff} = & \frac{m^2}{2} \phi^2 - \frac{1}{N_s^3 N_\tau} \sum_p \ln \left( 1 - \exp(\mu_\uparrow)(1 + \phi) \exp(iE) \right) \\ & - \frac{1}{N_s^3 N_\tau} \sum_p \ln \left( 1 - \exp(\mu_\downarrow)(1 + \phi) \exp(-iE) \right), \end{aligned} \quad (8)$$

where  $N_s$  is the lattice size of space, and  $E$  is energy or  $p_0$ . Now we compute the second term of the effective potential as

$$\begin{aligned}
\frac{\delta V_{eff}}{\delta \exp(\mu_\uparrow)} &= \frac{1}{N_s^3 N_\tau} \sum_p \frac{\exp(iE)}{1 - \exp(\mu_\uparrow)(1 + \phi) \exp(iE)} \\
&= \frac{1}{N_s^3 N_\tau} \sum_p \frac{1}{\exp(-iE) - \exp(\mu_\uparrow)(1 + \phi)} \\
&= \frac{1}{N_s^3 N_\tau} \sum_{\vec{p}} \oint_C dE \frac{1}{(\exp(-iE) - \exp(\mu_\uparrow)(1 + \phi))(\exp(iEN_\tau) + 1)} \frac{N_\tau}{2\pi} \\
&= \sum_{\vec{p}} \frac{1}{N_s^3 N_\tau} \frac{N_\tau}{2\pi} (-2\pi i) \frac{1}{-i \exp(\mu_\uparrow)(1 + \phi) \left( (\exp(\mu_\uparrow)(1 + \phi))^{-N_\tau} + 1 \right)} \\
&= - \frac{(\exp(\mu_\uparrow)(1 + \phi))^{N_\tau - 1}}{1 + (\exp(\mu_\uparrow)(1 + \phi))^{N_\tau}}.
\end{aligned}$$

Because we consider the anti-periodic boundary condition of the fermion field, we have

$$\exp(ip_i N^i) = -1, \quad p_i = \frac{2\pi(n_i + \frac{1}{2})}{N_i}, \quad n_i = -\left[\frac{N_i}{2}\right], \dots, 0, \dots, \left[\frac{N_i - 1}{2}\right]. \quad (9)$$

Hence, we sum over all energy modes, which is equivalent to doing complex integration with a contour  $C$ , which encloses the poles of  $\exp(iEN_\tau) + 1 = 0$ . If our contour encloses all poles of the integrand, then integration will vanish. Thus, we can obtain the fourth equality from the other pole. Then we can do integration to obtain dependence of  $\mu_\uparrow$  in the effective potential. The third term of the effective potential can be computed similarly. Thus, the effective potential is

$$V_{eff} = \frac{m^2}{2} \phi^2 - \frac{1}{N_\tau} \ln \left[ \left( 1 + (\exp(\mu_\uparrow)(1 + \phi))^{N_\tau} \right) \left( 1 + (\exp(\mu_\downarrow)(1 + \phi))^{N_\tau} \right) \right]. \quad (10)$$

If we consider  $n$ -sheet, then the effective potential becomes

$$\frac{m^2}{2} \phi^2 - \frac{1}{nN_\tau} \ln \left[ \left( 1 + (\exp(\mu_\uparrow)(1 + \phi))^{nN_\tau} \right) \left( 1 + (\exp(\mu_\downarrow)(1 + \phi))^{nN_\tau} \right) \right]. \quad (11)$$

The entropy is

$$\lim_{n \rightarrow 1} \left( - \frac{\partial}{\partial n} \ln \frac{Z_n}{Z^n} \right) = N_s^3 \ln((1 + \xi_1^{N_\tau})(1 + \xi_2^{N_\tau})) - N_s^3 N_\tau \frac{\ln \xi_1}{1 + \xi_1^{-N_\tau}} - N_s^3 N_\tau \frac{\ln \xi_2}{1 + \xi_2^{-N_\tau}}, \quad (12)$$

where  $Z_n$  is the partition function of the  $n$ -sheet and  $Z$  is the original partition function,  $\xi_1 \equiv \exp(\mu_\uparrow)(1 + \phi)$  and  $\xi_2 \equiv \exp(\mu_\downarrow)(1 + \phi)$ , and  $\phi$  satisfies

$$m^2\phi - \frac{1}{1 + \xi_1^{-N_\tau}} \frac{1}{1 + \phi} - \frac{1}{1 + \xi_2^{-N_\tau}} \frac{1}{1 + \phi} = 0. \quad (13)$$

When we take strong coupling limit ( $m \rightarrow 0$ ),  $\phi \rightarrow \infty$  and  $m^2\phi \rightarrow 0$ . Hence, the entropy vanishes under this limit for each temperature. This model may be trivial for the entanglement entropy because the entropy vanish because we do not have spatial derivative or we only have local on-site interacting terms when taking infinite strong coupling limit and fermion mass. The result supports that if we only have local on-site interacting terms under some limits, then the entanglement entropy should vanish. Thus, the entropy in this model only comes from thermal entropy, which gives volume law. The non-trivial point of this computation shows thermal entropy vanish at infinitely strong coupling limit. This result indicate that we do not have any physical degrees of freedom in infinitely strong coupling limit. This result also implies that the classification of phase structure is necessary to include topological field theory in lattice theories.

### 3 Entanglement Entropy of Two Dimensional Topological Field Theories

We first review the entanglement entropy of one-form Abelian gauge theory [13], and discuss lattice artifact in entanglement entropy of topological field theories from using a plaquette method and overlap formulation. We also find the entanglement entropy in two dimensional lattice topological field theory by using the plaquette method can be rewritten as the classical Shannon entropy as in one-form Abelian gauge theory. Thus, this property does not be modified by lattice artifact.

#### 3.1 Two Dimensional Abelian Gauge Theory

The partition function of two dimensional Abelian gauge theory is

$$Z_1 = \int DA \exp \left( -\frac{1}{2} \int d^2x E^2 \right), \quad (14)$$



where  $E \equiv \partial_0 A_1 - \partial_1 A_0 \equiv F_{01}$  is electric field,  $A_\mu$  is the gauge potential associated with the field strength  $F_{\mu\nu}$  or  $F_{01}$ , and we also denote the Greek indices as spacetime indices. We consider  $\partial_\mu A^\mu = 0$ , then we obtain  $A_\mu = -\frac{\pi k}{V} \epsilon_{\mu\nu} x^\nu$ , where we denote  $V$  is two dimensional area, and  $\epsilon_{01} = -\epsilon_{10} = -1$ . Thus, we obtain

$$Z_1 = \sum_k \exp\left(-\frac{2\pi^2 k^2}{V}\right), \quad Z_n = \sum_k \exp\left(-\frac{2\pi^2 k^2 n}{V}\right), \quad (15)$$

where  $Z_n$  is the partition function in  $n$ -sheet space. Then the entanglement entropy is

$$\begin{aligned} -\lim_{n \rightarrow 1} \frac{\partial}{\partial n} \frac{Z_n}{Z_1^n} &= -\lim_{n \rightarrow 1} \frac{\partial}{\partial n} \frac{\sum_k \exp\left(-\frac{2\pi^2 k^2 n}{V}\right)}{\left[\sum_i \exp\left(-\frac{2\pi^2 i^2}{V}\right)\right]^n} \\ &= \frac{\sum_k \frac{2\pi k^2}{V} \exp\left(-\frac{2\pi k^2}{V}\right)}{\sum_i \exp\left(-\frac{2\pi i^2}{V}\right)} + \ln \left[ \sum_k \exp\left(-\frac{2\pi k^2}{V}\right) \right]. \end{aligned} \quad (16)$$

Indeed, this entanglement entropy can be rewritten as classical Shannon entropy  $-\sum_k p_k \ln p_k$ , where

$$p_k \equiv \frac{\exp\left(-\frac{2\pi^2 k^2}{V}\right)}{\sum_i \exp\left(-\frac{2\pi i^2}{V}\right)}. \quad (17)$$

### 3.2 Lattice Artifact

Entanglement entropy of topological field theories only depends on topology of manifold. Thus, this is independent of coupling constant. On lattice, the entanglement entropy possibly vanishes in infinitely strong coupling constant for some theories on lattice, except for topological terms. Thus, entanglement entropy in topological field theories possibly plays an important role on lattice theories. Because lattice artifact comes from renormalization group flow or depends on coupling constant, entanglement entropy in topological field theories may not depend on lattice discretization. However, we consider

$$S_\theta = -\frac{i\theta}{2\pi} \int d^2 x F_{01}, \quad (18)$$

and find its different discretization to answer this question from a plaquette method and an overlap formulation, which preserves topological property on lattice. We also find the entanglement entropy from a plaquette method can be rewritten as the form of the classical Shannon entropy as in one-form Abelian gauge theory.

### 3.2.1 Plaquette Method

The lattice action of  $S_\theta$  from a plaquette method is

$$S_p = -\frac{\theta}{2\pi} \sum_p \ln(U_p), \quad (19)$$

where  $U_p$  is the product of the link variables  $U_\mu \equiv \exp(iA_\mu)$  around a plaquette. The lattice model needs to use a periodic boundary condition to preserve topological property on lattice. Then we compute the entanglement entropy on lattice from the plaquette method. We first define  $U_p \equiv \exp(if_{\mu\nu})$ , where  $-\pi < f_{\mu\nu} \leq \pi$ . Then the partition function [14] is

$$\begin{aligned} & \prod_{\mu\nu} \int_{-\pi}^{\pi} \frac{df_{\mu\nu}}{2\pi} \exp\left(\frac{i\theta}{2\pi} \sum_{\rho\sigma} f_{\rho\sigma}\right) \sum_n \delta\left(\sum_{\delta\gamma} f_{\delta\gamma} - 2\pi n\right) \\ &= \prod_{\mu\nu} \int_{-\pi}^{\pi} \frac{df_{\mu\nu}}{2\pi} \exp\left(\frac{i\theta}{2\pi} \sum_{\rho\sigma} f_{\rho\sigma}\right) \sum_m \exp\left(im \sum_{\delta\gamma} f_{\delta\gamma}\right) = \prod_{\mu\nu} \int_{-\pi}^{\pi} \frac{df_{\mu\nu}}{2\pi} \sum_m \exp\left(i\frac{\theta + 2\pi m}{2\pi} \sum_{\rho\sigma} f_{\rho\sigma}\right) \\ &= \sum_m \left[ \frac{2}{\theta + 2\pi m} \sin\left(\frac{\theta + 2\pi m}{2}\right) \right]^V. \end{aligned} \quad (20)$$

Then the entanglement entropy is

$$\begin{aligned} & -\lim_{n \rightarrow 1} \frac{\partial}{\partial n} \frac{Z_n}{Z_1^n} = -\lim_{n \rightarrow 1} \frac{\partial}{\partial n} \frac{\sum_m \left[ \frac{2}{\theta + 2\pi m} \sin\left(\frac{\theta + 2\pi m}{2}\right) \right]^{nV}}{\left[ \sum_i \left[ \frac{2}{\theta + 2\pi i} \sin\left(\frac{\theta + 2\pi i}{2}\right) \right]^V \right]^n} \\ &= \ln \left[ \sum_m \left[ \frac{2}{\theta + 2\pi m} \sin\left(\frac{\theta + 2\pi m}{2}\right) \right]^V \right] \\ & \quad - \frac{V}{\sum_m \left[ \frac{2}{\theta + 2\pi m} \sin\left(\frac{\theta + 2\pi m}{2}\right) \right]^V} \sum_i \left[ \frac{2}{\theta + 2\pi i} \sin\left(\frac{\theta + 2\pi i}{2}\right) \right]^V \ln \left[ \frac{2}{\theta + 2\pi i} \sin\left(\frac{\theta + 2\pi i}{2}\right) \right]. \end{aligned} \quad (21)$$

The entanglement entropy can also be rewritten as classical Shannon entropy  $-\sum_m p_m \ln p_m$ , where

$$p_m \equiv \frac{\left[ \frac{2}{\theta+2\pi m} \sin \left( \frac{\theta+2\pi m}{2} \right) \right]^V}{\sum_i \left[ \frac{2}{\theta+2\pi i} \sin \left( \frac{\theta+2\pi i}{2} \right) \right]^V}. \quad (22)$$

Thus, we guarantee the entanglement entropy in topology field theory can be rewritten as the classical Shannon entropy on lattice. This property possibly be protected by the topological property.

### 3.2.2 Overlap Formulation

Now we introduce the overlap formulation to preserve topological property on lattice, but this formulation does not guarantee to give same entanglement entropy as in the plaquette method. We construct the lattice theory by

$$\frac{1}{2} \text{Tr}(\gamma_5 D_{ov}), \quad (23)$$

where  $\gamma_5 \equiv i\gamma_1\gamma_2$ ,  $\gamma_1^2 = \gamma_2^2 = \gamma_5^2 = 1$ ,  $\gamma_1\gamma_2 = -\gamma_2\gamma_1$ ,  $\text{Tr}(\gamma_5) = 0$ , and  $D_{ov}$ , where we review the construction of overlap operator in Appendix. A. The overlap operator  $D_{ov}$  satisfies

$$D_{ov}\gamma_5 + \gamma_5 D_{ov} = D_{ov}\gamma_5 D_{ov}, \quad \gamma_5 D_{ov}\gamma_5 = D_{ov}^\dagger. \quad (24)$$

The properties also imply

$$D_{ov} + D_{ov}^\dagger = D_{ov}^\dagger D_{ov} = D_{ov} D_{ov}^\dagger. \quad (25)$$

Thus,  $D_{ov}$  is a normal operator, and its eigenvectors can form an orthogonal basis. The topological property is preserved on lattice can shown as

$$\begin{aligned} \frac{1}{2} \text{Tr}(\gamma_5 D_{ov}) &= -\frac{1}{2} \text{Tr} \left( \gamma_5 (2 - D_{ov}) \right) = -\frac{1}{2} \sum_{\lambda'} \left\langle \lambda' \left| \gamma_5 (2 - D_{ov}) \right| \lambda' \right\rangle = -\frac{1}{2} \sum_{\lambda'} (2 - \lambda') \langle \lambda' | \gamma_5 | \lambda' \rangle \\ &= n_- - n_+, \end{aligned} \quad (26)$$

where  $n_-$  is the number of negative eigenvalues (-1) of  $\gamma_5$  and  $n_+$  is the number of positive eigenvalues (+1) of  $\gamma_5$ . We also used  $D_{ov}|\lambda'\rangle = \lambda'|\lambda'\rangle$ . The last equality only comes from zero modes of  $D_{ov}$ . To explain the last equality, we introduce some properties of  $D_{ov}$ .

The eigenvalues of  $D_{ov}$  should be real or come in complex conjugate pairs. This is shown as

$$\begin{aligned}\det(D_{ov} - \lambda \cdot I) &= \det\left(\gamma_5^2(D_{ov} - \lambda \cdot I)\right) = \det\left(\gamma_5(D_{ov} - \lambda \cdot I)\gamma_5\right) = \det\left((D_{ov}^\dagger - \lambda \cdot I)\right) \\ &= \det\left((D_{ov} - \lambda^* \cdot I)\right)^*.\end{aligned}\quad (27)$$

Because we have

$$\langle\lambda'|D_{ov} + D_{ov}^\dagger|\lambda'\rangle = \langle\lambda'|D_{ov}^\dagger D_{ov}|\lambda'\rangle, \quad (28)$$

we can obtain  $\lambda'^* + \lambda' = \lambda'^* \lambda'$ . Then we use  $\lambda = x + iy$  to rewrite, and get  $(x-1)^2 + y^2 = 1$ . We can also express  $\lambda'$  as

$$\begin{aligned}\lambda' &\equiv 1 - e^{i\phi}, \quad \phi \in (-\pi, \pi], \\ \frac{1}{\lambda'} &= \frac{1}{1 - e^{i\phi}} = \frac{1}{1 - \cos\phi - i\sin\phi} = \frac{1 - \cos\phi + i\sin\phi}{2 - 2\cos\phi} = \frac{1}{2} + i\frac{\sin\phi}{2(1 - \cos\phi)}.\end{aligned}\quad (29)$$

Now we can show that zero modes of  $D_{ov}$  is the eigenstates of  $\gamma_5$  as

$$D_{ov}|0\rangle = 0 \Rightarrow \gamma_5 D_{ov}|0\rangle = 0 \Rightarrow 0 = (D_{ov}\gamma_5 D_{ov} - D_{ov}\gamma_5)|0\rangle = -D_{ov}\gamma_5|0\rangle \Rightarrow \gamma_5|0\rangle = \pm|0\rangle, \quad (30)$$

in where we used  $\gamma_5^2 = 1$  in the last arrow. Now we show  $\langle\lambda'|\gamma_5|\lambda'\rangle = 0$  unless  $\lambda' \in R$  as

$$\lambda'\langle\lambda'|\gamma_5|\lambda'\rangle = \langle\lambda'|\gamma_5 D_{ov}|\lambda'\rangle = \langle\lambda'|D_{ov}^\dagger \gamma_5|\lambda'\rangle = \lambda'^* \langle\lambda'|\gamma_5|\lambda'\rangle \Rightarrow (\text{Im}\lambda') \cdot \langle\lambda'|\gamma_5|\lambda'\rangle = 0. \quad (31)$$

From the above properties of  $D_{ov}$ , we show

$$-\frac{1}{2} \sum_{\lambda'} (2 - \lambda') \langle\lambda'|\gamma_5|\lambda'\rangle = n_- - n_+ \quad (32)$$

because we only need to consider  $\lambda'=0, 2$ , and the contribution vanishes in the case of  $\lambda' = 2$ .

When we consider continuum limit of the lattice theory, then the continuum theory is

$$-i\theta \lim_{M \rightarrow \infty} \text{Tr} \left[ \gamma_5 \exp \left( \frac{D_c^2}{M^2} \right) \right], \quad (33)$$

where  $D_c \equiv \gamma_\mu (\partial_\mu + iA_\mu)$ , and  $D_c$  satisfies  $D_c \gamma_5 + \gamma_5 D_c = 0$  and  $\gamma_5 D_c \gamma_5 = D_c^\dagger$ . Because  $D_c = D_L - D_L^\dagger$ , where  $D_L = D_c(1 + \gamma_5)/2$  and  $D_L^\dagger = \gamma_5 D_L \gamma_5$ , we can find all non-zero eigenvalues of  $D_L^\dagger D_L$   $D_L D_L^\dagger$  are paired. For example,

$$D_L^\dagger D_L \phi = \lambda \phi \Rightarrow D_L D_L^\dagger D_L \phi = \lambda D_L \phi. \quad (34)$$

We can see  $D_L \phi$  is the eigenfunction of  $D_L D_L^\dagger$  with the same eigenvalue as the eigenstates of  $D_L^\dagger D_L$ . The continuum theory can be rewritten as

$$-i\theta \left[ \text{Tr} \left[ \exp \left( -\frac{D_L^\dagger D_L}{M^2} \right) \right] - \text{Tr} \left[ \exp \left( -\frac{D_L D_L^\dagger}{M^2} \right) \right] \right] = -i\theta \text{Tr} \left[ \gamma_5 \exp \left( \frac{D_c^2}{M^2} \right) \right]. \quad (35)$$

Because all non-zero eigenvalues of  $D_L^\dagger D_L$  and  $D_L D_L^\dagger$  are paired, the non-zero eigenvalues will be canceled. Therefore, we can take  $M^2 \rightarrow \infty$  to do computation because the continuum theory should be independent of  $M^2$ . Now we show the continuum theory is the expected theory as

$$\begin{aligned} -i\theta \lim_{M \rightarrow \infty} \text{Tr} \left[ \gamma_5 \exp \left( \frac{D_c^2}{M^2} \right) \right] &= -i\theta \lim_{M \rightarrow \infty} \int d^2x \frac{d^2p}{(2\pi)^2} \text{Tr} \left[ \gamma_5 \exp \left( -\frac{(p+A)^2}{M^2} + \frac{i}{2M^2} F_{\mu\nu} \gamma_\mu \gamma_\nu \right) \right] \\ &= -i\theta \lim_{M \rightarrow \infty} \int d^2x \text{Tr} \left( \frac{i}{2M^2} \gamma_5 F_{\mu\nu} \gamma_\mu \gamma_\nu \right) \int \frac{d^2p}{(2\pi)^2} e^{-\frac{p^2}{M^2}} \\ &= -i\theta \int d^2x \left( -\frac{2}{M^2} F_{01} \cdot \frac{M^2}{4\pi} \right) = \frac{-i\theta}{2\pi} \int d^2x F_{01}. \end{aligned} \quad (36)$$

Hence, a suitable lattice action from the overlap formulation is

$$S_{ov} = \frac{-i\theta}{2} \text{Tr}(\gamma_5 D_{ov}). \quad (37)$$

Finally, we compare two lattice methods. Two lattice methods can preserve topological property without taking continuum limit. Naively, entanglement entropy in topological field theory should not depend on coupling constant. Thus, we possibly expect entanglement entropy in topological field theory does not suffer from lattice artifact.

Indeed, the lattice action can be constructed from the plaquette method and overlap formulation, and they all give the same action when we take continuum limit. This does not guarantee we can have the same topological effects in different lattice formulation with the same lattice size and lattice spacing. The lattice simulation shows the plaquette method can give larger fluctuation of winding numbers than the overlap formulation in [15]. Our conclusion is the entanglement entropy of topological field theory on lattice needs to be careful for different definitions of lattice action, or lattice artifact.

## 4 Entanglement Entropy in Quantum Gravity Theory

We discuss the entanglement entropy in quantum gravity. Because quantum gravity has some expected properties, but it is still hard to understand from the first principle. Thus, we first use two dimensional Einstein-Hilbert action, which is reviewed in Appendix. B, to compute the entanglement entropy. This theory is topological theory and conformal field theory, and it also appears in string theory to give different topology of moduli space. Hence, two dimensional Einstein-Hilbert theory is a suitable model to know theoretical properties of quantum gravity in different coupling regions. We also assume area law is a necessary condition for the entanglement entropy, then we argue that translational invariance is needed.

### 4.1 Two Dimensional Einstein-Hilbert Theory

The action of the two dimensional Einstein-Hilbert theory is

$$S_{EH} = -\frac{1}{16\pi G} \int d^2x \sqrt{\det g_{\mu\nu}} R = -\frac{1}{4G} \chi, \quad (38)$$

where  $\chi \equiv 2 - 2g$ ,  $g$  is genus,  $G$  is Newton constant, and

$$R_{\mu\nu} \equiv \partial_\delta \Gamma_{\nu\mu}^\delta - \partial_\nu \Gamma_{\delta\mu}^\delta + \Gamma_{\delta\lambda}^\delta \Gamma_{\nu\mu}^\lambda - \Gamma_{\nu\lambda}^\delta \Gamma_{\delta\mu}^\lambda, \quad \Gamma_{\nu\delta}^\mu \equiv \frac{1}{2} g^{\mu\lambda} \left( \partial_\delta g_{\lambda\nu} + \partial_\nu g_{\lambda\delta} - \partial_\lambda g_{\nu\delta} \right), \quad (39)$$

$$R \equiv g^{\mu\nu} R_{\mu\nu}. \quad (40)$$

We compute the entanglement entropy by  $n$ -sheet method [16] as

$$\begin{aligned}
-\lim_{n \rightarrow 1} \frac{\partial}{\partial n} \frac{Z_n}{Z_1^n} &= -\lim_{n \rightarrow 1} \frac{\partial}{\partial n} \frac{\sum_{\chi'} e^{\frac{1}{4G}(n\chi' - 2N(n-1))}}{(\sum_{\chi} e^{\frac{\chi}{4G}})^n} = -\frac{\sum_{\chi'} e^{\frac{1}{4G}\chi'} \frac{1}{4G}(\chi' - 2N)}{\sum_{\chi} e^{\frac{\chi}{4G}}} + \ln \left( \sum_{\chi} e^{\frac{\chi}{4G}} \right) \\
&= \ln \left( \sum_{\chi} e^{\frac{\chi}{4G}} \right) - \frac{1}{4G} \frac{\sum_{\chi'} e^{\frac{\chi'}{4G}} \chi'}{\sum_{\chi} e^{\frac{\chi}{4G}}} + \frac{\langle N \rangle}{2G},
\end{aligned} \tag{41}$$

where  $2N$  is number of ramification points,  $\langle N \rangle$  is the expectation value of  $N$  and  $\chi$  is the Euler number. The partition function is computed by summing over all Riemann surfaces (different genus and numbers of ramification points). If we define

$$p_i \equiv \frac{e^{\frac{1}{4G}\chi_i}}{\sum_{\chi} e^{\frac{1}{4G}\chi}}, \tag{42}$$

then

$$-\sum_i p_i \ln p_i = \ln \left( \sum_{\chi} e^{\frac{\chi}{4G}} \right) - \frac{1}{4G} \frac{\sum_{\chi'} e^{\frac{\chi'}{4G}} \chi'}{\sum_{\chi} e^{\frac{\chi}{4G}}}. \tag{43}$$

Thus, the entanglement entropy can be rewritten as

$$-\sum_i p_i \ln p_i + \frac{\langle N \rangle}{2G}. \tag{44}$$

Indeed, the expression is very interesting because the result can be seen as sum of classical entanglement and quantum entanglement, and it is also a quantum extension of [4], in which they find two dimensional entropy has the same and unique form if a theory has translational invariance, Poincaré symmetry, causality and finite entanglement entropy. We review this result in Appendix. C. If we only consider classical background and spherical geometry, then different numbers of ramification points correspond to different numbers of intervals because two dimensional Einstein-Hilbert action also has conformal symmetry [17]. Then we also get the consistent result as in [4]. For two dimensional theories, the entangling surface is a point. Thus, we can think analogous area quantity is number of ramification points, and each interval has two ramification points. The quantum entanglement is  $\frac{2\langle N \rangle}{4G}$  in the entanglement entropy of two dimensional Einstein-Hilbert action. In higher dimensions, the analogue term is  $\frac{\langle A \rangle}{4G}$ , where  $A$  is a co-dimensional two

surface. This possibly motivates us to think quantum gravity has area operator  $2\hat{N}$  or  $\hat{A}$ . The sum over all ramification points in the path integral is also equivalent with summing over all classical configuration in an entangling surface. Thus, we think that this also supports that the area term possibly comes from centers of a Hilbert space in an entangling surface. Hence, the result of two dimensional Einstein-Hilbert action possibly inspires us to find non-negative constant plus non-negative area terms for the entanglement entropy from physical principles, or define a suitable area operator to explore quantum gravity.

## 4.2 Non-Volume Law of the Entanglement Entropy

We want to use dimensional analysis to argue entanglement entropy is impossible to have volume law when a theory is translational invariant, and satisfies subadditivity at zero temperature in an infinite size system without mass scale. We first use translational invariance, then the entanglement entropy must depend on translational invariant quantities. For example, size of a system. We also have subadditivity law [2] as

$$S_A + S_B \geq S_{AB} \quad (45)$$

if  $\rho_{AB}$  is a density matrix in  $H_{AB}$ , isomorphic to  $\oplus_i H_A^i \otimes H_B^i$ . Then we can show the density of the entanglement entropy should be finite when spatial volume goes to infinity as

$$\begin{aligned} & \frac{S_A(k_1, k_2, \dots, k_n)}{k_1 k_2 \dots k_n} < \infty, \\ & S_A(a_1, a_2, \dots, a_n) = S_A(p_1 k_1 + q_1, p_2 k_2 + q_2, \dots, p_n k_n + q_n) \\ & \leq p_1 p_2 \dots p_n S_A(k_1, k_2, \dots, k_n) + (\text{low numbers of } p_i), \\ & \frac{S_A(a_1, a_2, \dots, a_n)}{a_1 a_2 \dots a_n} \leq \frac{p_1 k_1}{a_1} \frac{p_2 k_2}{a_2} \dots \frac{p_n k_n}{a_n} \frac{S_A(k_1, k_2, \dots, k_n)}{k_1 k_2 \dots k_n} + \dots, \\ & \lim_{a_1, a_2, \dots, a_n \rightarrow \infty} \frac{S_A(a_1, a_2, \dots, a_n)}{a_1 a_2 \dots a_n} \leq \frac{S_A(k_1, k_2, \dots, k_n)}{k_1 k_2 \dots k_n} < \infty \end{aligned} \quad (46)$$

in where we used  $a_i \equiv p_i k_i + q_i$ ,  $0 \leq q_i \leq k_i - 1$ . Therefore, we obtain the density of the entanglement entropy should be finite if a theory is translational invariant, and satisfies subadditivity at zero temperature in an infinite size of a system without mass scale. If our system is at zero temperature in an infinite size system without mass scale, then we only have regularization parameter and the side length of a sub-physical system with



a unit of length. If we have volume term in the entanglement entropy, then this term should be proportional to  $V/\epsilon^{D-1}$ , where  $V$  is the size of a system,  $\epsilon$  is the regularization parameter and  $D$  is total dimensions of spacetime. Thus, it is easy to find the density of the entanglement entropy should be divergent. We argue non-volume law of the entanglement entropy at zero temperature in an infinite size system without mass scale should need translational invariant, and satisfy subadditivity law, which can be shown from time translational invariant in a quantum system. Because we expect quantum gravity should have area law of the entanglement entropy, our discussion possibly implies translational invariance is a necessary condition in quantum gravity theory from ruling out volume law. Although we cannot include mass scale in our discussion, strongly coupled conformal is expected to describe weakly coupled bulk gravity theory, and the mass term will break conformal symmetry or scale invariance in a theory.

## 5 Two Dimensional Conformal Field Theories

In the strong coupled conformal field theory, minimum surface of weakly coupled gravity theory can give universal terms of the entanglement entropy in conformal field theory. This means universal terms of the entanglement entropy in conformal field theory can be rewritten in terms of geometrical quantities. A choice of an entangling surface in the entanglement entropy comes from a decomposition of a Hilbert space, but this should not modify geometry of a space. Thus, we want to argue conformal field theory in strong coupling regions possibly does not depend on a choice of an entangling surface, or universal terms of the entanglement entropy is uniquely determined, unaffected by a choice of the entangling surface. We first consider two dimensional conformal field theory. In this theory, we can use mathematical methods to determine universal terms of the entanglement entropy for single interval uniquely, and this method can also be extended to some cases of multiple intervals. We discuss generic multiple intervals in two dimensional conformal field theory from geometric methods. Finally, we consider two dimensional  $CP^{N-1}$  model. This model also has conformal symmetry. Thus, we can compute the entanglement entropy exactly in the large  $N$  limit and  $AdS$  background, and can have unique form of the entanglement entropy for single interval. If we have holographic dual, then the entanglement entropy should be gotten from strongly coupled  $CFT_1$ , and discuss the consistent understanding as in two dimensional  $CP^{N-1}$  model.

## 5.1 Two Dimensional Conformal Field Theory

We first use translational invariance and strong subadditivity [2, 7, 18, 19], then we find

$$S_A(l_A) + S_B(l_B) \geq S_{A \cup B}(l_{A \cup B}) + S_{A \cap B}(l_{A \cap B}). \quad (47)$$

Then we consider Poincaré symmetry or boost symmetry to constraint the length of the systems for one interval in two dimensional quantum field theories. We have four points  $a_1 = (0, 0)$ ,  $a_2 = (0, c_1)$ ,  $a_3 = (d_2, c_1 + d_1)$  and  $a_4 = (d_2 + b_1, c_1 + d_1)$ , in where we use null coordinates  $u = t + x$ ,  $v = t - x$ , to define the size  $\sqrt{uv}$  of systems  $A$  ( $\overrightarrow{a_1 a_3} = \sqrt{d_2(c_1 + d_1)} \equiv l_A$ ),  $B$  ( $\overrightarrow{a_2 a_4} = \sqrt{d_1(b_1 + d_2)} \equiv l_B$ ),  $A \cup B$  ( $\overrightarrow{a_1 a_4} = \sqrt{(d_2 + b_1)(c_1 + d_1)} \equiv l_{A \cup B}$ ) and  $A \cap B$  ( $\overrightarrow{a_2 a_3} = \sqrt{d_1 d_2} \equiv l_{A \cap B}$ ), and we assumed  $(u_i, v_i) \equiv a_i$ . Thus, we find

$$l_{A \cup B} \cdot l_{A \cap B} = l_A l_B = \sqrt{d_1 d_2 (d_2 + b_1) (c_1 + d_1)}, \quad \frac{l_A}{l_{A \cup B}} = \frac{l_{A \cap B}}{l_B} \equiv \frac{1}{\lambda}, \quad (48)$$

where  $\lambda \geq 1$ . This interesting relation to imply

$$S_A(l_A) - S_{A \cap B}(l_{A \cap B}) \geq S_{A \cup B}(l_{A \cup B}) - S_B(l_B) = S_{A \cup B}(\lambda l_A) - S_B(\lambda l_{A \cap B}). \quad (49)$$

Finally, we use modular transformation to restrict the form of the entanglement entropy. Then we consider a  $SL(2, C)$  transformation

$$x \rightarrow \frac{ax + b}{cx + d}, \quad ad - bc = 1, \quad (50)$$

which is a conformal transformation because scaling, inversion, and translation still preserves angles. The parameters  $a$ ,  $b$ ,  $c$  and  $d$  are constants. Therefore, we can construct an invariant quantity under the conformal transformation as

$$\begin{aligned} u_2 - u_3 &\rightarrow \frac{au_2 + b}{cu_2 + d} - \frac{au_3 + b}{cu_3 + d} = \frac{(au_2 + b)(cu_3 + d) - (au_3 + b)(cu_2 + d)}{(cu_2 + d)(cu_3 + d)} \\ &= \frac{(ad - bc)(u_2 - u_3)}{(cu_2 + d)(cu_3 + d)} = \frac{u_2 - u_3}{(cu_2 + d)(cu_3 + d)}, \\ \frac{(u_2 - u_3)(u_1 - u_4)}{(u_1 - u_3)(u_2 - u_4)} &\rightarrow \frac{(u_2 - u_3)(u_1 - u_4)}{(u_1 - u_3)(u_2 - u_4)}, \quad \frac{(v_2 - v_3)(v_1 - v_4)}{(v_1 - v_3)(v_2 - v_4)} \rightarrow \frac{(v_2 - v_3)(v_1 - v_4)}{(v_1 - v_3)(v_2 - v_4)}. \end{aligned} \quad (51)$$

We also know  $F \equiv S_A(l_A) + S_B(l_B) - S_{A \cap B}(l_{A \cap B}) - S_{A \cup B}(l_{A \cup B})$  is invariant under the modular transformation. From

$$\begin{aligned}
& F\left(\frac{(u_2 - u_3)(u_1 - u_4)}{(u_1 - u_3)(u_2 - u_4)}, \frac{(v_2 - v_3)(v_1 - v_4)}{(v_1 - v_3)(v_2 - v_4)}\right) \\
&= S_A\left(\sqrt{(u_1 - u_3)(v_1 - v_3)}\right) + S_B\left(\sqrt{(u_2 - u_4)(v_2 - v_4)}\right) \\
&- S_{A \cap B}\left(\sqrt{(u_2 - u_3)(v_2 - v_3)}\right) - S_{A \cup B}\left(\sqrt{(u_1 - u_4)(v_1 - v_4)}\right), \tag{52}
\end{aligned}$$

then we can obtain the entanglement entropy

$$S_A = k_1 \ln l_A + k_2, \tag{53}$$

where  $k_1$  and  $k_2$  are constants, because we can derive  $f = k_1 \ln x + k_2$  for  $g(x^2 y^2) = f(x) + f(y)$  as

$$g(x^2 y^2) = f(x) + f(y) \Rightarrow 2xy^2 \frac{dg(x^2 y^2)}{dx} = \frac{df(x)}{dx} \Rightarrow \frac{2}{x} \frac{dg}{dx}(1) \equiv \frac{k_1}{x} = \frac{df(x)}{dx} \Rightarrow f(x) = k_1 \ln x + k_2. \tag{54}$$

Thus, the form of the entanglement entropy is determined from translational invariance, strong subadditivity, boost symmetry and conformal symmetry [6]. The determination of the entanglement entropy from mathematical methods can also include different choices of an entangling surface because we only impose symmetry to constraint the form of the entanglement entropy, and do not restrict a choice of an entangling surface. The result does not show that we exclude that universal terms of the entanglement entropy does not modify from a choice of an entangling surface because we cannot restrict the value of the universal coefficient. To know the dependence of a choice of centers in the entanglement entropy, we need to know other entanglement physical quantities, which can be related to universal terms of the entanglement entropy. We already found an example to support the mutual information does not depend on a choice of centers in two dimensional conformal field theory for multiple intervals [7]. Now we can show that universal terms in the mutual information does not depend on a choice of centers for one interval. We show that the universal terms of the entanglement entropy is not affect by a choice of an entangling surface. If the entanglement entropy is

$$S_A = k_1 \ln l_A + k_2, \tag{55}$$

then the mutual information is

$$k_1 \ln \frac{l_A l_B}{l_{A \cup B}} + k_2 \quad (56)$$

if each region is single interval. We use the same method as in [7] to show the mutual information is independent of a choice of centers, and know  $k_1$  should not be affected by a choice of an entangling surface. Hence, the universal terms for one interval in two dimensional conformal field theory should be unique. Now we mention the logic of showing uniqueness of the mutual information. We first insert boundary state in an entangling surface. Then the  $n$ -sheet partition function of a cylinder for considering ground state is given by

$$Z_n = \langle a_1^{(n)} | \exp \left( \frac{\ell}{n} \frac{c}{12} \right) | a_2^{(n)} \rangle, \quad (57)$$

where  $|a_{1,2}^{(n)}\rangle$  are the boundary states from the cutoff circle,  $\ell = \ln(l_A/\epsilon)^2$ , and  $c$  is the center charge. We can use the conformal mapping

$$w = \ln \frac{z - z_1}{z - z_2}, \quad (58)$$

to get the single interval. The Rényi entropy for ground state in a single interval case is

$$S_n = \left(1 + \frac{1}{n}\right) \frac{c}{6} \left(\log \frac{l_A}{\epsilon}\right) + \frac{1}{1-n} (s(a_1^{(n)}) - ns(a_1^{(1)}) + s^*(a_2^{(n)}) - ns^*(a_2^{(1)})), \quad (59)$$

where  $s(a_1^{(n)}) = \log \langle a_1^{(n)} | 0 \rangle$  is the boundary entropy. Then we can find boundary entropy should disappear in the mutual information when we insert the boundary states to consider different choices of an entangling surface. Therefore, we can conclude the universal term of the entanglement entropy must be  $c/3$  for one interval without any modification from a choice of an entangling surface. This method can also be extended to some cases of multiple intervals from [7]. To give a generic discussion, we choose a geometric method to discuss because we expect geometry of a theory should not be modified by a choice of an entangling surface. Thus, the universal terms of the entanglement entropy in a strongly coupled conformal field theory possibly be unique if we have correspondence between geometry and the entanglement entropy.

In general, a higher genus Riemann surface can be obtained from the quotient of a complex plane by a discrete subgroup of  $SL(2, C)$  as

$$\mathcal{M} = C'/\Sigma, \quad (60)$$

where  $C'$  is the complex plane with some bad points (fixed points of  $\Sigma$ ) removed. The  $AdS_3$  geometry is

$$ds^2 = \frac{1}{z^2}(dz^2 - dt^2 + dx^2), \quad (61)$$

then we can take  $z \rightarrow 0$  to get a complex plane

$$ds^2 \rightarrow \frac{1}{z^2}(-dt^2 + dx^2) = -\frac{dudv}{z^2}. \quad (62)$$

When the theory has a gravity dual, the same quotient can be extended to the bulk and it determines the bulk geometry.

Now considering the map from the complex  $w$ -plane  $C$  before the quotient to  $\mathcal{M}$

$$\pi_S : C \rightarrow \mathcal{M}.$$

The function  $w(z) = \pi_S^{-1}(z)$  is multi-valued on  $\mathcal{M}$  in the sense that after a closed loop, then it can be a different point in  $C$  by the action of an element in  $\Sigma$ . The map can be expressed in the following form

$$w(z) = \frac{\psi_1(z)}{\psi_2(z)}, \quad (63)$$

where  $\psi_{1,2}$  are the two linearly independent solutions of the equation

$$\psi''(z) + \frac{1}{2}T_{zz}(z)\psi(z) = 0, \quad (64)$$

where the stress tensor  $T_{zz} \equiv \sum_{i=1,2,\dots,2N} \left( \Delta/(z - z_i)^2 + p_i/(z - z_i) \right)$ , where  $\Delta = (n^2 - 1)/(2n^2)$ ,  $p_i$  is called accessory parameters,  $N$  is the number of intervals, follows from the transformation of  $T_{ww} = 0$ , *i.e.*, the Schwarzian derivative

$$T_{zz}(z) = \{\pi_S^{-1}(z), z\} \equiv \frac{w'''(z)}{w'(z)} - \frac{3}{2} \left( \frac{w''(z)}{w'(z)} \right)^2. \quad (65)$$

It is straightforward to check that (65) is satisfied for such a  $w(z)$ . We note that

$$w'(z) = \frac{\psi_1'\psi_2 - \psi_1\psi_2'}{\psi_2^2} \equiv \frac{W(z)}{\psi_2^2(z)}, \quad W'(z) = \psi_1''\psi_2 - \psi_1\psi_2'' = -\frac{1}{2}T_{zz}\psi_1\psi_2 + \frac{1}{2}T_{zz}\psi_1\psi_2 = 0,$$

and hence

$$w''(z) = \frac{-2W(z)\psi_2'(z)}{\psi_2^3(z)}, \quad w'''(z) = \frac{-2W(z)[\psi_2''(z)\psi_2(z) - 3(\psi_2'(z))^2]}{\psi_2^4(z)},$$

and then

$$\frac{w'''(z)}{w'(z)} - \frac{3}{2} \left( \frac{w''(z)}{w'(z)} \right)^2 = \frac{-2[\psi_2''(z)\psi_2(z) - 3(\psi_2'(z))^2]}{\psi_2^2(z)} - \frac{6\psi_2'^2(z)}{\psi_2^2(z)} = \frac{-2\psi_2''(z)\psi_2(z)}{\psi_2^2(z)} = T_{zz}(z).$$

Recall that  $w(z)$  has nontrivial monodromy, and this implies the following action of  $\Sigma$  on  $\psi_{1,2}$  as

$$(\psi_1, \psi_2) \rightarrow (\psi_1, \psi_2)M(C), \quad M(C) \equiv \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \Sigma. \quad (66)$$

The nontrivial monodromy (and also  $\Sigma$ ) form a representation of the fundamental group of  $\mathcal{M}$ .

The total number of independent cycles is  $2(n-1)(N-1)$  and they are the generators of the fundamental group. Now considering the cycles  $C_M$  in the  $z$ -plane (instead of  $\mathcal{M}$ ) that encircle an even number of points  $z_i$ . We can pick a set  $\Gamma$  of  $N$  independent non-intersecting cycles of this type

$$\Gamma = \{C_M : M = 1, \dots, N\}. \quad (67)$$

There can be many different choices of  $\Gamma$  for each  $N$ . We can label the  $\mathcal{N}_N$  configurations of  $N$  cycles  $C_M$  by  $\Gamma_\gamma$  and they form a set  $\mathcal{T}_N$

$$\mathcal{T}_N = \{\Gamma_\gamma : \gamma = 1, \dots, \mathcal{N}_N\}. \quad (68)$$

The number  $\mathcal{N}_N$  can be obtained from the recursive relation  $\mathcal{N}_N = 3\mathcal{N}_{N-1} - \mathcal{N}_{N-2}$ .

The  $2N$  parameters  $p_i$  can be fixed by imposing trivial monodromy around the  $N$  cycles  $C_M$ , each of which corresponds to a cycle contractible in the bulk solution of handle body. Each different configuration  $\Gamma_\gamma$  gives a set of parameters  $p_i^\gamma$ . Notice that for a surface of  $g = (n-1)(N-1)$ , we will need  $(n-1)(N-1)$  non-intersecting independent cycles ( $A$ -cycles) to be contractible for the handle body. They can be generated by applying replica symmetry to the  $N-1$  independent cycles among  $C_M$  <sup>2</sup>

$$\{R^m(C_M) : m = 0, \dots, n-1; M = 1, \dots, N-1\}, \quad (69)$$

where  $R$  map a cycle to the next replica copy. Notice again that these cycles satisfy the constraint  $\sum_{m=0}^{n-1} [R^m(C_M)]$  is trivial and hence the total number of independent ones is  $(N-1)(n-1)$ .

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<sup>2</sup>One of the  $C_M$  is contractible at  $z = \infty$  and hence trivial on  $\mathcal{M}$ .

Due to the trivial monodromy around the  $A$ -cycles, the Schottky group  $\Sigma$  as a representation of the fundamental group can only be generated by the other half ( $B$ -cycles) of the generators  $\{L_m, m = 1, \dots, (n-1)(N-1)\}$ . As a result the fundamental domain of  $C'/\Sigma$  is given by the exterior of the pairs of non-intersecting circles related by  $L_m(C_m) = \tilde{C}_m$ .

Now we want to reproduce the entanglement entropy for one interval in two dimensional conformal field theory by using this geometric approach. We first solve accessory parameters. We consider replica limit as

$$\delta_n = n - 1 \rightarrow 0, \quad p_i \rightarrow \rho_i \delta_n, \quad \Delta \rightarrow \delta_n, \quad (70)$$

where  $\rho_i$  are constants. We solve

$$\frac{d}{dz} \begin{pmatrix} \psi \\ \psi' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2}T_{zz} & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix} \equiv H \begin{pmatrix} \psi \\ \psi' \end{pmatrix}. \quad (71)$$

For convenience, we also define

$$M \equiv \exp \left( \int dz H \right) \equiv M_0 M_I, \quad (72)$$

in where  $M_0$  comes from a leading order solution when we use a perturbation method to solve. Under the replica limit, we can obtain

$$\begin{aligned} \frac{d}{dz} \begin{pmatrix} \psi \\ \psi' \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\frac{\delta_n}{2}T_1 & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix} \equiv (H_0 + \delta_n H_1) \begin{pmatrix} \psi \\ \psi' \end{pmatrix}, \\ T_1 &\equiv \sum_i \left( \frac{1}{(z - z_i)^2} + \frac{\rho_i}{(z - z_i)} \right), \end{aligned} \quad (73)$$

where  $H_0$  comes from the leading order of the replica limit, and  $H_1$  comes from next leading order of the replica limit. Then we take zeroth order solution

$$\psi_0 = az + b, \quad (74)$$

where  $a$  and  $b$  are constants, to obtain

$$\begin{pmatrix} \psi \\ \psi' \end{pmatrix} = \begin{pmatrix} 1 & z - z_0 \\ 0 & 1 \end{pmatrix} M_I \begin{pmatrix} \psi \\ \psi' \end{pmatrix}, \quad (75)$$

Then it is easy to find

$$\begin{aligned}\frac{d}{dz}M_I &= -M_0^{-1}\left(\frac{d}{dz}M_0\right)M_0^{-1}M + M_0^{-1}\frac{d}{dz}M = -M_0^{-1}H_0M_0M_0^{-1}M + M_0^{-1}\left(H_0 + \delta_n H_1\right)M \\ &= M_0^{-1}\delta_n H_1 M_0 M_I.\end{aligned}\quad (76)$$

Therefore, we can get

$$M_I \approx 1 + \frac{\delta_n}{2} \int dz \begin{pmatrix} (z - z_0) & (z - z_0)^2 \\ -1 & -(z - z_0) \end{pmatrix} T_1. \quad (77)$$

We can choose the integration range near  $z_0$ , then we should find the vanishing of integration result. We obtain three independent conditions as

$$\sum_{i=1}^{2N} \rho_i = 0, \quad \sum_{i=1}^{2N} (\rho_i z_i + 1) = 0, \quad \sum_{i=1}^{2N} (\rho_i z_i^2 + 2z_i) = 0. \quad (78)$$

The conditions are also  $SL(2, R)$  invariant as in

$$\begin{aligned}z_i &\rightarrow \frac{Az_i + B}{Cz_i + D}, \quad AD - BC = 1, \\ \rho_i &\rightarrow \rho_i (Cz_i + D)^2 + 2C(Cz_i + D) = \rho_i C^2 z_i^2 + (2CD\rho_i + 2C^2)z_i + (\rho_i D^2 + 2CD) \\ \sum_{i=1}^{2N} \rho_i &\rightarrow \sum_{i=1}^{2N} \left( -2z_i C^2 - 2CD + 2C^2 z_i + 2CD \right) = 0, \\ \sum_{i=1}^{2N} \rho_i z_i &\rightarrow \sum_{i=1}^{2N} (Az_i + B)(Cz_i + D) \left( \rho_i + 2\frac{C}{Cz_i + D} \right) \\ &= \sum_{i=1}^{2N} \rho_i (Az_i + B)(Cz_i + D) + 2C(Az_i + B) \\ &= \sum_{i=1}^{2N} AC\rho_i z_i^2 + (AD\rho_i + BC\rho_i + 2AC)z_i + (BD\rho_i + 2BC) \\ &= \sum_{i=1}^{2N} \left( -2ACz_i - AD - BC + 2ACz_i + 2BC \right) = -\sum_{i=1}^{2N} 1, \\ -2\sum_{i=1}^{2N} z_i &\rightarrow -2\sum_{i=1}^{2N} \frac{Az_i + B}{Cz_i + D} = -2\sum_{i=1}^{2N} \left( \frac{A}{C} + \frac{B - \frac{AD}{C}}{Cz_i + D} \right) = -\frac{4AN}{C} - 2\sum_{i=1}^{2N} \frac{B - \frac{AD}{C}}{Cz_i + D} \\ &= -4\frac{AN}{C} - 2\left( B - \frac{AD}{C} \right) \sum_{i=1}^{2N} \frac{1}{Cz_i + D},\end{aligned}\quad (79)$$



$$\begin{aligned}
\sum_{i=1}^{2N} p_i z_i^2 &\rightarrow \sum_{i=1}^{2N} (Az_i + B)^2 \left( \rho_i + 2 \frac{C}{Cz_i + D} \right) \\
&= \sum_{i=1}^{2N} \left( A^2 \rho_i z_i^2 + 2AB \rho_i z_i + B^2 \rho_i \right) + (Az_i + B)(2C) \frac{Az_i + B}{Cz_i + D} \\
&= -2A^2 \left( \sum_{i=1}^{2N} z_i \right) - 4ABN + 2BC \sum_{i=1}^{2N} \frac{Az_i + B}{Cz_i + D} + 2AC \sum_{i=1}^{2N} \frac{Az_i + B}{Cz_i + D} z_i \\
&= -2A^2 \left( \sum_{i=1}^{2N} z_i \right) - 4ABN + 2BC \sum_{i=1}^{2N} \left( \frac{A}{C} + \frac{B - \frac{AD}{C}}{Cz_i + D} \right) + 2AC \sum_{i=1}^{2N} \left( \frac{A}{C} + \frac{B - \frac{AD}{C}}{Cz_i + D} \right) z_i \\
&= -2A^2 \left( \sum_{i=1}^{2N} z_i \right) - 4ABN + 4ABN - 2B \sum_{i=1}^{2N} \frac{1}{Cz_i + D} + 2A^2 \sum_{i=1}^{2N} z_i - 2A \sum_{i=1}^{2N} \frac{z_i}{Cz_i + D} \\
&= -2B \sum_{i=1}^{2N} \frac{1}{Cz_i + D} - \frac{4AN}{C} + \frac{2AD}{C} \sum_{i=1}^{2N} \frac{1}{Cz_i + D} \\
&= -4 \frac{AN}{C} - 2 \left( B - \frac{AD}{C} \right) \sum_{i=1}^{2N} \frac{1}{Cz_i + D}. \tag{80}
\end{aligned}$$

We can solve conditions to obtain

$$\rho_i = -\frac{2}{z_i - z_j}, \quad \rho_j = -\frac{2}{z_j - z_i} \tag{81}$$

for pairs  $(z_i, z_j) \in P^\gamma$ , where  $P^\gamma = \{(z_i, z_j)_K; K = 1, 2, \dots, N\}$  for  $N$  intervals. Then we use a formula to connect accessory parameters and entanglement entropy as

$$\frac{\partial S_n}{\partial z_i} = -\frac{k_1 n}{2(n-1)} p_i, \tag{82}$$

where  $S_n$  is the Rényi entropy, and  $p_i \approx \delta_n \rho_i$  when  $\delta_n \rightarrow 0$ . If we consider  $N = 1$ , then we obtain

$$S_A = S_1 = k_1 \ln |z_1 - z_2| + k_2. \tag{83}$$

Therefore, we can use the geometric construction to reproduce the results of one interval. Then we can also consider multiple intervals as

$$S_A = \min_{\gamma} \left( k_1^\gamma \sum_{(z_i, z_j) \in P_\gamma} \ln |z_i - z_j| + k_2^\gamma \right). \tag{84}$$

If we take the same regularization parameters for each interval, then we can find a consistent result as in two dimensional finite entropy for regularization dependent terms. When we take regularization parameters to be small and finite, the dominant term is a regularization dependent term, and these terms are finite under the limit. Due to the consistency, the approach may give a correct universal terms of the entanglement entropy. Because the construction can come from geometry, we expect the universal terms of the entanglement entropy should be unique for  $N$  intervals.

## 5.2 Two Dimensional $CP^{N-1}$ Model

The continuum theory for  $CP^{N-1}$  model, which is reviewed in the Appendix. D, is

$$S_{cp} = \beta N \int d^2x \left( \partial_\mu z_i^* \partial_\mu z_i + (z_i^* \partial_\mu z_i)(z_j^* \partial_\mu z_j) \right), \quad (85)$$

where  $z_i(x)$  is an  $N$  component complex field satisfying  $z_i^* z_i = 1$ ,  $\beta N \equiv 1/g$ , and  $g$  is a coupling constant. This model can approach to free theory in the large  $N$  limit. Thus, we can consider bulk scalar field to find holographic dual to strongly coupled one dimensional conformal field theory [9]. This theory is also a conformal field theory so we can compute the entanglement entropy in the large  $N$  limit [20], and the entanglement entropy for one interval is also unique. The entanglement entropy in  $AdS_2$  and large  $N$  limit should be half of the entanglement entropy in flat background, and proportional to  $N$ . The entanglement entropy in one dimensional strongly conformal field theory should be defined by tracing over particles rather than tracing over space. We consider a box with a periodic boundary condition, and we also take large volume limit and dilute limit as  $V \gg N \gg 1$ . We interpret  $N$  as numbers of particles in strongly coupled one dimensional conformal field theory. The density of particles is

$$\frac{N}{V} = \int \frac{d^D k}{(2\pi)^D} f_k, \quad (86)$$

and the entanglement entropy is

$$S_A = - \sum_k f_k \ln f_k = -V \int \frac{d^D k}{(2\pi)^D} f_k \ln f_k. \quad (87)$$

If  $f_k$  is proportional to  $1/V$  or we consider dilute gas approximation, then the entanglement entropy is proportional to  $N \ln V$ . The form of the entanglement entropy is similar

to the entanglement entropy for  $AdS_2$  and single interval in two dimensional  $CP^{N-1}$  model. Then the Sachdev-Ye-Kitaev models (SYK model) also gives a similar behavior of the entanglement entropy [21], which is proportional to the numbers of degrees of freedom in a sub-system. The choice of an entangling surface in the holographic dual possibly be the choice of the value for the potential on the boundary of the box. When we consider strong coupling limit in one dimensional conformal field theory, the potential behavior on the boundary of the box should not be important. The choice of an entangling surface in two dimensional conformal field theory [8] comes from inserting boundary states in entangling surfaces so our guess possibly gives a consistent thought. Thus, we argue  $AdS_2/CFT_1$  possibly does not have dependence on a choice of an entangling surface. We can also directly understand this point from the bulk theory. Because we also show that the universal terms of the entanglement entropy for single interval is unique, the universal terms of the entanglement entropy in two dimensional  $CP^{N-1}$  model in  $AdS_2$  background for single interval or some multiple intervals cases does not depend on a choice of an entangling surface. Thus, the argument can be consistent from studying the boundary theory or bulk theory.

Now we discuss lattice  $CP^{N-1}$  model. The lattice model can be written from putting link variables as

$$S_{lcpg} = -\beta N \sum_{x, \hat{\mu}} (z_{x+\hat{\mu}}^* \cdot z_x) U_{\mu}^*(x) + (z_x^* \cdot z_{x+\hat{\mu}}) U_{\mu}(x) \quad (88)$$

Thus, this seems a choice of an entangling surface may affect the entanglement entropy. We should remember a choice of an entangling surface comes from ultraviolet scale [3]. The continuum limit in the lattice  $CP^{N-1}$  model is in the weak coupling region and the large  $N$  limit so the link variables becomes auxiliary fields. The non-dynamical fields are not problematic to give a non-tensor product decomposition of a Hilbert space [7]. From the lattice point of view, we can also get the consistent understanding with the continuum theory for holographic dual.

## 6 Conclusion

We study various approaches or problems of the entanglement entropy in a strong coupling region. The first study is to consider the non-relativistic four fermion model with

spin imbalance on lattice. We compute the entanglement entropy from setting infinite fermion mass and taking infinite strong coupling constant to give zero entropy. This means we do not have any dynamical degrees of freedom on the non-relativistic fermion model in the infinite strong coupling region and infinite fermion mass as in the results of strong coupling expansion in the lattice Yang-Mills gauge theory [2]. Thus, this implies that we need to find topological quantities to classify the phase structure in strong coupling regions for some lattice theories. If a theory does not have entropy in the infinite strong coupling limit for trivial topology, then considering non-trivial topology is one way. We compute the entanglement entropy in two dimensional Abelian gauge theory, and two dimensional theta term to know theoretical properties of the entanglement entropy. The result of the entanglement entropy can be rewritten as the classical Shannon entropy without suffering from the lattice artifact. Thus, this property possibly be protected by topological property. We also discuss the lattice artifact for the entanglement entropy in topological field theory from the plaquette method and overlap formulation. Topological field theories do not depend on coupling constants so we may expect entanglement entropy in topological field theory should not suffer from lattice artifact. However, we can use the plaquette and overlap formulation to preserve topological property on lattice, but they only give the same result in continuum theory. Thus, the lattice artifact of the entanglement entropy in topological field theory cannot be avoided, and this problem also lets us to be careful about the larger fluctuation of the topological charge, winding number or topological number.

We also compute the entanglement entropy in two dimensional Einstein gravity theory, which gives different topology of moduli space in string theory, and is also topological and conformal field theories, from summing over all different numbers of ramification points and genus. Then the entanglement entropy is the sum of classical Shannon entropy and the analogous co-dimensional two surface terms (proportional to expectation values of numbers of ramification points). Therefore, we expect this study possibly implies quantum gravity needs an area operator. The path integral of two dimensional gravity theory also gives insights of the area terms possibly come from centers of an entangling surface or area operator is a center in the Hilbert space of quantum gravity theory because the area law comes from summing over ramification points. From the result, we strongly believe that the entanglement entropy in quantum gravity should have area law. Thus, we use translational invariant and subadditivity, which can be shown by unitary

or time translational invariant, in an infinite size system at zero temperature without mass scale to rule out the volume law of the entanglement entropy. Hence, this possibly shows quantum gravity needs translational invariance. Because our analysis does not use Poincaré symmetry, quantum gravity is defined on discrete variables is still possible.

Finally, we use symmetry principles [6] and uniqueness of the mutual information [7] to show the universal terms of entanglement entropy for single interval in two dimensional conformal field theory should be unique, and this result can also be extended to some cases of multiple intervals. For a single interval, we also use a geometric method [5] to determine the entanglement entropy in two dimensional conformal field theory. The geometric method can also be extended to generic multiple intervals, and give the consistent result as in two dimensional finite entropy [4]. Because we do not expect geometry in a theory will be modified by choosing different entangling surfaces. Thus, we argue universal terms of the entanglement entropy in two dimensional conformal field theory in a strongly coupled region is not modified by a choice of an entangling surface. We also discuss the dependence of a choice of an entangling surface of the entanglement entropy in two dimensional  $CP^{N-1}$  via holographic dual.

The dominant effect of the entanglement entropy in a strong coupling region on lattice comes from non-trivial topology under some limits. Thus, the non-trivial topology in the entanglement entropy should be easily observed in this environment. We are also interested in knowing whether we use entanglement entropy to classify the phase structure in strongly coupled QCD model.

The computation of the entanglement entropy in the QCD model must need the Monte-Carlo method to work. The lattice computation in the QCD model spend too much time so we may start two dimensional quantum electrodynamics (QED) with the topological theta term. We can numerically compute two dimensional QED model with the topological theta term from a reweighting method to do simulation. The main problem is different constructions of topological field theories will give different answers on lattice. To give a clear study, we may need to fix topology to numerically compute the entanglement entropy to study the phase structure in topological field theories.

Quantum gravity does not have many clues to be probed now. We expect area law of the entanglement entropy may be interesting to give insights to us. We are interested in what principles to give area law of the entanglement entropy, and we are also interested in finding candidates of quantum gravity on lattice from getting area law

of the entanglement entropy.

Strongly coupled conformal field theory is expected to describe weakly coupled perturbative quantum gravity, and a choice of an entangling surface should come from ultra-violet scale so strongly coupled conformal field theory is expected not to rely on a choice of an entangling surface. Thus, an interesting and important extension of our work is to find generic understanding of the entanglement entropy in higher dimensional conformal field theories. This should give more clear clues to us in quantum gravity and holographic duality.

We usually expect that the mutual information is proportional to center charge in conformal field theory. In four dimensional Abelian gauge theory gives a counter example to us. The mutual information is only proportional to bulk central charge. The boundary central charge is canceled in the mutual information. The definition of the entanglement entropy in gauge theories needs to sacrifice quantum fluctuation of an entangling surface to give new information to boundaries or entangling surfaces. We can think the strongly coupled conformal field theory only stores information in the bulk, then after flowing to weakly coupled theory, some information of bulk flow to boundary. Thus, we do not see boundary central charge in the mutual information. From the point of view of holographic principle, we expect the holographic dual theory [22] needs to store information in the bulk or on the boundary totally because we do not think physics of higher dimensions can be deduced from lower dimensional physics. Thus, this may give an interpretation why we should use strongly coupled conformal field theory to dual to weakly coupled bulk gravity theory. This is an interesting open question. We still do not have concrete computation to support the physical picture.

To prove uniqueness of the universal terms of the entanglement entropy in higher dimensional conformal field theories [23], we cannot use the same method to show because the mutual information possibly does not count all center charges. The proof may be fine for the strongly coupled conformal field theories because we expect strongly coupled conformal field theory does not have boundary center charge in the universal terms of the entanglement entropy. Our paper should give a starting point in this direction.

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## A Review of Overlap Operator

We construct the overlap operator in this section. We first introduce Wilson fermion. The continuum theory is

$$\bar{\psi}(\gamma_\mu(\partial_\mu + iA_\mu) + m)\psi. \quad (89)$$

The naive discretization of this theory is given by

$$\bar{\psi}_x D_{nw}(U)_{x,y} \psi_y, \quad (90)$$

where

$$D_{nw}(U)_{x,y} \equiv m\delta_{x,y} + \frac{1}{2} \sum_{\mu} \left( \gamma_\mu U_\mu(x) \delta_{x+\hat{\mu},y} - \gamma_\mu U_\mu^\dagger(x - \hat{\mu}) \delta_{x-\hat{\mu},y} \right). \quad U_\mu(x) \equiv e^{iA_\mu}. \quad (91)$$

Then the propagator in lattice fermion theory of this naive discretization is

$$\frac{1}{m + i \sum_{\mu} \gamma_\mu \sin(p_\mu)}. \quad (92)$$

This propagator has more poles than continuum theory, and the unphysical poles do not decouple from continuum limit. Therefore, the naive discretization is failed. To rescue this problem, we introduce

$$\frac{1}{2} \sum_{\mu} \left( 2\delta_{x,y} - U_\mu(x) \delta_{x+\hat{\mu},y} - U_\mu^\dagger(x - \hat{\mu}) \delta_{x-\hat{\mu},y} \right) \quad (93)$$

to the lattice action. This new term vanishes when we take continuum limit because the prefactor of this new term is  $a/2$ , where  $a$  is lattice spacing. Because we assume  $a = 1$ , lattice spacing does not appear in our theory. This new term is equivalent to

$$-\frac{1}{2} \partial_\mu \partial_\mu \quad (94)$$

in continuum. The propagator becomes

$$\frac{1}{m + i \sum_{\mu} \gamma_{\mu} \sin(p_{\mu}) + \sum_{\nu} (1 - \cos(p_{\nu}))} \quad (95)$$

when we introduce the new term. All non-physical poles will decouple from continuum limit, and the physical pole is not affected from the new term. If we take  $m = 0$ , then we have two choices of each component of poles  $p_{\mu} = 0, \pi$ . Each component of  $p_{\mu} = \pi$  has  $1/a$  mass term. When we take continuum limit, then infinite mass helps us to decouple the non-physical poles. If we consider all  $p_{\mu} = 0$ , then the new term does not have modification of the propagator. This lattice discretization does not have exact chiral symmetry on lattice. Thus, we introduce the overlap operator ( $D_{ov}$ ) to restore chiral symmetry on lattice. If the lattice theory is given by

$$\bar{\psi} D_{ov} \psi, \quad (96)$$

and the chiral transformation of the fermion is given by

$$\psi' \equiv \psi \exp \left[ i\alpha \gamma_5 \left( 1 - \frac{D_{ov}}{2} \right) \right], \quad \bar{\psi}' = \bar{\psi} \exp \left[ i\alpha \left( 1 - \frac{D_{ov}}{2} \right) \gamma_5 \right], \quad (97)$$

where  $\alpha$  is a constant then

$$\begin{aligned} \bar{\psi}' D_{ov} \psi' &= \bar{\psi} \exp \left[ i\alpha \left( 1 - \frac{D_{ov}}{2} \right) \gamma_5 \right] D_{ov} \exp \left[ i\alpha \gamma_5 \left( 1 - \frac{D_{ov}}{2} \right) \right] \psi \\ &= \bar{\psi} \exp \left[ i\alpha \left( 1 - \frac{D_{ov}}{2} \right) \gamma_5 \right] \exp \left[ -i\alpha \left( 1 - \frac{D_{ov}}{2} \right) \gamma_5 \right] D_{ov} \psi = \bar{\psi} D_{ov} \psi. \end{aligned} \quad (98)$$

We used

$$D_{ov} \gamma_5 + \gamma_5 D_{ov} = D_{ov} \gamma_5 D_{ov} \quad (99)$$

in the second equality. The chiral transformation is antisymmetric because we treat  $\psi$  and  $\bar{\psi}$  are independent fields on lattice. Thus, the exact chiral symmetry is restored on lattice. Now we formulate the overlap operator. The overlap operator is

$$D_{ov} = 1 + \gamma_5 H (H^2)^{-\frac{1}{2}} \equiv 1 + \gamma_5 \text{sign}(H), \quad (100)$$



where  $H \equiv \gamma_5(D_w - 1)$  and

$$\begin{aligned} D_w(U)_{x,y} &\equiv m\delta_{x,y} + \frac{1}{2} \sum_{\mu} \left( \gamma_{\mu} U_{\mu}(x) \delta_{x+\hat{\mu},y} - \gamma_{\mu} U_{\mu}^{\dagger}(x - \hat{\mu}) \delta_{x-\hat{\mu},y} \right) \\ &\quad + \frac{1}{2} \sum_{\mu} \left( 2\delta_{x,y} - U_{\mu}(x) \delta_{x+\hat{\mu},y} - U_{\mu}^{\dagger}(x - \hat{\mu}) \delta_{x-\hat{\mu},y} \right). \end{aligned} \quad (101)$$

This overlap operator satisfies

$$\begin{aligned} D_{ov} D_{ov}^{\dagger} &= (1 + \gamma_5 \text{sign}(H)) (1 + \text{sign}(H) \gamma_5) = 1 + \gamma_5 \text{sign}(H) + \text{sign}(H) \gamma_5 + 1 \\ &= D_{ov} + D_{ov}^{\dagger}, \end{aligned} \quad (102)$$

and

$$\gamma_5 D_{ov} \gamma_5 = 1 + \text{sign}(H) \gamma_5 = D_{ov}^{\dagger}. \quad (103)$$

Hence, these properties imply

$$D_{ov} \gamma_5 D_{ov} \gamma_5 = D_{ov} + \gamma_5 D_{ov} \gamma_5, \Rightarrow D_{ov} \gamma_5 D_{ov} = D_{ov} \gamma_5 + \gamma_5 D_{ov}. \quad (104)$$

Now we can define a suitable overlap operator to preserve exact chiral symmetry on lattice.

## B Review of Two Dimensional Einstein-Hilbert Action

We review the two dimensional Einstein-Hilbert theory from the Gauss-Bonnet theorem. We first introduce useful knowledge and definition in curves, three dimensional surface, the first fundamental theorem, and curvature of surfaces. Then we show the Gauss-Bonnet theorem, and the Gauss and Codazzi-Mainardi equations. Finally, we write the result of the two dimensional Einstein-Hilbert action.

### B.1 Curves in the Plane and in the Space

We list all basic properties and definitions of curves in this section for self-contained reading.

**Definition 1.** A parametrized curve in  $R^n$  is a map  $\gamma : (\alpha, \beta) \equiv \{t \in R | \alpha < t < \beta\} \rightarrow R^n$  for some  $\alpha$  and  $\beta$  with  $-\infty < \alpha \leq \beta < \infty$ .

**Definition 2.** If  $\gamma$  is a parametrized curve, its first derivative  $\dot{\gamma}(t)$  is the tangent vector of the parametrized curve  $\gamma$  at the point  $\gamma(t)$ .

**Definition 3.** The arc-length of a curve  $\gamma$  starting at a point  $\gamma(t_0)$  is the function  $s(t)$

$$s(t) = \int_{t_0}^t \|\dot{\gamma}(u)\| du. \quad (105)$$

**Definition 4.** If  $\gamma : (\alpha, \beta) \rightarrow R^n$  is a parametrized curve, its speed at the point  $\gamma(t)$  is  $\|\dot{\gamma}(t)\|$ , and  $\gamma$  is said to be a unit-speed curve if  $\dot{\gamma}(t)$  is a unit vector  $\forall t \in (\alpha, \beta)$ .

**Proposition 1.** Let  $\vec{n}(t)$  be a unit vector that is a smooth function of a parameter  $t$ . Then we have

$$\dot{\vec{n}} \cdot \vec{n} = 0 \quad (106)$$

$\forall t$ . This also implies  $\dot{\vec{n}}$  is zero or perpendicular to  $\vec{n}(t) \forall t$ .

Thus, if  $\gamma$  is a unit-speed curve, then  $\ddot{\gamma}$  is zero or perpendicular to  $\dot{\gamma}$ .

**Definition 5.** A parametrized curve  $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \rightarrow R^n$  is a reparametrization of a parametrized curve  $\gamma : (\alpha, \beta) \rightarrow R^n$  if there is a smooth bijective map  $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ , the inverse map is also smooth, and

$$\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t})) \quad \forall t \in (\tilde{\alpha}, \tilde{\beta}). \quad (107)$$

**Definition 6.** A point  $\gamma(t)$  of a parametrized curve  $\gamma$  is called a regular point if  $\dot{\gamma} \neq 0$ . Otherwise,  $\gamma(t)$  is a singular point of  $\gamma$ . A curve is regular if all points of the curve are regular.

**Proposition 2.** If  $\gamma$  is a regular curve, its arc-length  $s$ , starting at any points of the regular curve  $\gamma$ , is a smooth function of  $t$ .

*Proof.* Because the arc-length  $s$  is a differentiable function of  $t$ , we have

$$\frac{ds}{dt} = \|\dot{\gamma}(t)\|. \quad (108)$$

Then  $\gamma(t) \equiv (u(t), v(t))$ , where  $u(t)$  and  $v(t)$  are smooth functions. Thus, we get

$$\frac{ds}{dt} = \sqrt{\dot{u}^2 + \dot{v}^2}. \quad (109)$$

The function  $\sqrt{\dot{u}^2 + \dot{v}^2}$  is a smooth function on the open interval  $0 < \dot{u}, \dot{v} < \infty$  so the arc-length is also smooth. ■

**Proposition 3.** *A parametrized curve  $\gamma$  has a unit-speed reparametrization  $\tilde{\gamma}$  if and only if it is regular.*

*Proof.* Suppose that a parametrized curve  $\gamma : (\alpha, \beta) \rightarrow R^n$  has a unit-speed reparametrization  $\tilde{\gamma}$  with a map  $\phi : \tilde{t} \rightarrow t$ . We have  $\tilde{\gamma}(\tilde{t}) = \gamma(t)$ , and

$$\frac{d\tilde{\gamma}(\tilde{t})}{d\tilde{t}} = \frac{d\gamma(t)}{dt} \frac{dt}{d\tilde{t}} \Rightarrow \left\| \frac{d\tilde{\gamma}(\tilde{t})}{d\tilde{t}} \right\| = \left\| \frac{d\gamma(t)}{dt} \right\| \cdot \left| \frac{dt}{d\tilde{t}} \right|. \quad (110)$$

Because  $\tilde{\gamma}$  is unit-speed, we obtain  $\|d\tilde{\gamma}(\tilde{t})/d\tilde{t}\| = 1$ . Hence,  $d\gamma(t)/dt$  cannot be zero.

Now we suppose the tangent vector  $d\gamma(t)/dt$  is not zero for each point of the parametrized curve  $\gamma$ . By the Proposition 2, the arc-length of the regular parametrized curve  $\gamma$  is also a smooth function of  $t$ . We also know  $s : (\alpha, \beta) \rightarrow R$ , where its image is an open interval  $(\tilde{\alpha}, \tilde{\beta})$ , is injective so we can get the inverse map  $s^{-1}$  is also smooth. Now we choose  $\tilde{\gamma}$  be the corresponding reparametrization of  $\gamma$  as

$$\tilde{\gamma}(s(t)) = \gamma(t). \quad (111)$$

Then we get

$$\frac{d\tilde{\gamma}(s)}{ds} \frac{ds(t)}{dt} = \frac{d\gamma(t)}{dt} \Rightarrow \left\| \frac{d\tilde{\gamma}(s)}{ds} \right\| \left| \frac{ds}{dt} \right| = \left\| \frac{d\gamma(t)}{dt} \right\| = \frac{ds}{dt} \Rightarrow \left\| \frac{d\tilde{\gamma}}{ds} \right\| = 1. \quad (112)$$

■

**Definition 7.** *A parametrized curve  $\gamma : R \rightarrow R^n$  is a smooth curve, and  $T \in R$ , then  $\gamma$  is  $T$ -periodic if*

$$\gamma(t + T) = \gamma(t) \quad \forall t \in R. \quad (113)$$

*If  $\gamma$  is not a constant and is  $T$ -periodic for some  $T \neq 0$ , then  $\gamma$  is closed.*

**Definition 8.** *The period of a closed curve  $\gamma$  is the smallest positive number  $T$ .*

**Definition 9.** *A curve is said to have a self-intersection at a point  $\vec{p}$  of the curve if there exists parameters  $a \neq b$  to give*

1.  $\gamma(a) = \gamma(b) = \vec{p}$ ,
2. if  $\gamma$  is closed with period  $T$ , then  $a - b$  is not an integer multiple of  $T$ .

## B.2 Surfaces in Three Dimensions

We give useful definitions and theorems for convenience.

**Definition 10.** *A simple closed curve in  $R^2$  is a closed curve in  $R^2$  that has no self-intersections. We also define the simple closed curve is positively-oriented if the signed unit normal of  $\gamma$  or the unit vector by rotating the tangent vector  $\dot{\gamma}$  anticlockwise by  $\pi/2$  points into  $\text{int}(\gamma)$  for each point of  $\gamma$ .*

**Theorem 1.** *The complement of the image of a simple closed curve  $\gamma$  in  $R^2$  (The points are in  $R^2$  and they are not in the image of  $\gamma$ .) is the disjoint union of two subsets of  $R^2$  by  $\text{int}(\gamma)$  and  $\text{ext}(\gamma)$ , which has following properties as*

1.  $\text{int}(\gamma)$  is bounded such that the interior region is contained inside a circle of sufficiently large radius,
2.  $\text{ext}(\gamma)$  is unbounded,
3. Both of the regions  $\text{int}(\gamma)$  and  $\text{ext}(\gamma)$  are connected such that they have property two points in regions can be joined by a curve contained in the region, and any curves joins a point of  $\text{int}(\gamma)$  to a point of  $\text{ext}(\gamma)$  must cross the simple closed curve  $\gamma$  in  $R^2$ .

**Definition 11.** *A subset  $S$  of  $R^3$  is a surface if there is an open set  $U$  in  $R^2$  for every points  $p \in S$ , and an open set  $W$  in  $R^3$  containing  $p$  such that  $S \cap W$  is homeomorphic to  $U$ . A subset of a surface  $S$  of the form  $S \cap W$  is called an open subset of  $S$ . A homeomorphism  $\vec{\sigma} : U \rightarrow S \cap W$  is called a surface patch or parametrization of the open subset  $S \cap W$  of  $S$ . A collection of such surface patches whose surface images contains the whole of  $S$  is called an atlas of  $S$ .*

**Definition 12.** *A surface is orientable if there exists an atlas for the surface if we have a transition map between two surface patches in the atlas, then determinant of the transition map is larger than zero.*

**Definition 13.** A tangent vector of a surface  $S$  at a point  $\vec{p} \in S$  is the tangent vector of a curve in the surface  $S$  passing through the point  $\vec{p}$ . The tangent space  $T_p S$  of the surface  $S$  at the point  $\vec{p}$  is the set of all tangent vectors of the surface  $S$  at the point  $\vec{p}$ .

**Proposition 4.** If we have an orientable surface, then there is a smooth choice of an unit normal at each point of the orientable surface. We can take the standard unit normal of any surface patch in an atlas of the orientable surface. The standard unit normal is defined as

$$\vec{N}_\sigma \equiv \frac{\vec{\sigma}_u \times \vec{\sigma}_v}{\|\vec{\sigma}_u \times \vec{\sigma}_v\|}, \quad (114)$$

where  $\vec{\sigma}$  is a surface patch of the orientable surface, and  $\vec{\sigma}_u \equiv \partial \vec{\sigma} / \partial u$ .

**Theorem 2.** For each genus (number of holes in a surface)  $g \geq 0$ ,  $T_g$  (A surface has  $g$  holes.) has an atlas making it a smooth surface. Moreover, each compact surface is diffeomorphic to one of the  $T_g$ .

**Corollary 1.** Each compact surface is orientable.

*Proof.* Each surface  $T_g$  has a bounded interior and an unbounded exterior. Therefore, we can choose the unit normal at each point of the surface  $T_g$  to point into the exterior region. This implies  $T_g$  is orientable because we find a smooth choice of the unit normal. Because each compact surface is also diffeomorphic to one of the compact surface  $T_g$ , compact surfaces should be orientable. ■

### B.3 The First Fundamental Form

**Definition 14.** The first fundamental form of a surface patch  $\vec{\sigma}(u, v)$  is

$$\begin{aligned} \langle \vec{v}, \vec{v} \rangle &= \lambda^2 \langle \vec{\sigma}_u, \vec{\sigma}_u \rangle + 2\lambda\mu \langle \vec{\sigma}_u, \vec{\sigma}_v \rangle + \mu^2 \langle \vec{\sigma}_v, \vec{\sigma}_v \rangle = E\lambda^2 + 2F\lambda\mu + G\mu^2 \\ &= E du(\vec{v})^2 + 2F du(\vec{v}) dv(\vec{v}) + G dv(\vec{v})^2 = E du^2 + 2F du dv + G dv^2, \end{aligned} \quad (115)$$

where  $E = \|\vec{\sigma}_u\|^2$ ,  $F = \vec{\sigma}_u \cdot \vec{\sigma}_v$  and  $G = \|\vec{\sigma}_v\|^2$ , if we define maps  $du : T_p S \rightarrow R$  and  $dv : T_p S \rightarrow R$  by

$$du(\vec{v}) = \lambda \in R, \quad dv(\vec{v}) = \mu \in R, \quad \text{if } \vec{v} = \lambda \vec{\sigma}_u + \mu \vec{\sigma}_v. \quad (116)$$

**Definition 15.** The area  $A_\sigma(R)$  of the region  $R$  for a surface patch  $\vec{\sigma} : U \rightarrow R^3$  corresponding to the region  $R \subseteq U$  is

$$A_\sigma(R) \equiv \int_R dudv \|\vec{\sigma}_u \times \vec{\sigma}_v\|. \quad (117)$$

**Proposition 5.** When the first fundamental form is  $Edu^2 + 2Fdudv + Gdv^2$ ,  $\|\vec{\sigma}_u \times \vec{\sigma}_v\| = (EG - F^2)^{1/2}$ .

*Proof.* We first compute

$$\|\vec{\sigma}_u \times \vec{\sigma}_v\|^2 = (\vec{\sigma}_u \times \vec{\sigma}_v) \cdot (\vec{\sigma}_u \times \vec{\sigma}_v) = (\vec{\sigma}_u \cdot \vec{\sigma}_u)(\vec{\sigma}_v \cdot \vec{\sigma}_v) - (\vec{\sigma}_u \cdot \vec{\sigma}_v)^2 = EG - F^2, \quad (118)$$

then we get  $\|\vec{\sigma}_u \times \vec{\sigma}_v\| = (EG - F^2)^{1/2}$ . ■

## B.4 Curvature of Surfaces

We discuss the second fundamental form, and normal geodesic curvatures in this section.

### B.4.1 The Second Fundamental Form

When a surface patch  $\vec{\sigma}(u, v)$  at a point  $(u, v)$  move to the surface patch  $\vec{\sigma}(u + \Delta u, v + \Delta v)$  at a point  $(u + \Delta u, v + \Delta v)$ , the distance for moving away parallel to a unit normal  $\vec{N}$  is

$$\begin{aligned} & (\vec{\sigma}(u + \Delta u, v + \Delta v) - \vec{\sigma}(u, v)) \cdot \vec{N} \\ & \approx \left[ \vec{\sigma}_u \Delta u + \vec{\sigma}_v \Delta v + \frac{1}{2} \left( \vec{\sigma}_{uu} (\Delta u)^2 + 2\vec{\sigma}_{uv} \Delta u \Delta v + \vec{\sigma}_{vv} (\Delta v)^2 \right) \right] \cdot \vec{N} \\ & = \frac{1}{2} \left( \vec{\sigma}_{uu} (\Delta u)^2 + 2\vec{\sigma}_{uv} \Delta u \Delta v + \vec{\sigma}_{vv} (\Delta v)^2 \right) \cdot \vec{N}. \end{aligned} \quad (119)$$

Thus, we define the second fundamental form of the surface patch  $\vec{\sigma}$  is

$$Ldu^2 + 2Mdudv + Ndv^2, \quad (120)$$

where

$$L = \vec{\sigma}_{uu} \cdot \vec{N}, \quad M = \vec{\sigma}_{uv} \cdot \vec{N}, \quad N = \vec{\sigma}_{vv} \cdot \vec{N}. \quad (121)$$

### B.4.2 The Gauss and Weingarten Maps

**Definition 16.** The Weingarten map  $W_{\vec{p},S}$  of a surface  $S$  at a point  $\vec{p}$  is defined by

$$W_{\vec{p},S} \equiv -D_{\vec{p}}G, \quad (122)$$

where  $D_{\vec{p}}G$  is the Jacobian matrix when we use the matrix to represent the Weingarten map  $W_{\vec{p},S}$ ,  $G$  is a Gauss map from the surface  $S$  to a unit sphere  $S^2$ , the Weingarten map  $W_{\vec{p},S}$  can map each point  $\vec{p}$  to the unit normal  $N_{\vec{p}} \in S^2$  of the surface  $S$ , and the Weingarten map is also a map from  $T_pS \rightarrow T_{G(\vec{p})}S^2$ . Because the Weingarten map  $W_{\vec{p},S}$  is just a linear map,  $W_{\vec{p},S} : T_pS \rightarrow T_pS$ .

**Lemma 1.** We have

$$\vec{N}_u \cdot \vec{\sigma}_u = -\vec{\sigma}_{uu} \cdot \vec{N}, \quad \vec{N}_u \cdot \vec{\sigma}_v = \vec{N}_v \cdot \vec{\sigma}_u = -\vec{\sigma}_{uv} \cdot \vec{N}, \quad \vec{N}_v \cdot \vec{\sigma}_v = -\vec{\sigma}_{vv} \cdot \vec{N} \quad (123)$$

for a surface patch  $\sigma$  with a standard unit normal.

*Proof.* Because we have

$$\vec{N} \cdot \vec{\sigma}_u = \vec{N} \cdot \vec{\sigma}_v = 0, \quad (124)$$

we get

$$\vec{N}_u \cdot \vec{\sigma}_u = -\vec{N} \cdot \vec{\sigma}_{uu}, \quad \vec{N}_v \cdot \vec{\sigma}_u = -\vec{N} \cdot \vec{\sigma}_{uv}, \quad \vec{N}_u \cdot \vec{\sigma}_v = -\vec{N} \cdot \vec{\sigma}_{uv}, \quad \vec{N}_v \cdot \vec{\sigma}_v = -\vec{N} \cdot \vec{\sigma}_{vv}. \quad (125)$$

■

**Proposition 6.** If we have the second fundamental form of a surface patch  $\vec{\sigma}$

$$Ldu^2 + 2Mdudv + Ndv^2, \quad (126)$$

then we can get

$$\langle \vec{v}, \vec{w} \rangle \equiv W(\vec{v}) \cdot \vec{w} = Ldu(\vec{v})du(\vec{w}) + M(du(\vec{v})dv(\vec{w}) + du(\vec{w})dv(\vec{v})) + Ndv(\vec{v})dv(\vec{w}), \quad (127)$$

where  $\vec{v}, \vec{w} \in T_pS$ .

*Proof.* Because we know

$$du(\vec{\sigma}_u) = dv(\vec{\sigma}_v) = 1, \quad du(\vec{\sigma}_v) = dv(\vec{\sigma}_u) = 0, \quad (128)$$

we can get

$$\begin{aligned} \langle \vec{\sigma}_u, \vec{\sigma}_u \rangle &= -\vec{N}_u \cdot \vec{\sigma}_u = L, & \langle \vec{\sigma}_u, \vec{\sigma}_v \rangle &= -\vec{N}_u \cdot \vec{\sigma}_v = M, \\ \langle \vec{\sigma}_v, \vec{\sigma}_u \rangle &= -\vec{N}_v \cdot \vec{\sigma}_u = M, & \langle \vec{\sigma}_v, \vec{\sigma}_v \rangle &= -\vec{N}_v \cdot \vec{\sigma}_v = N. \end{aligned} \quad (129)$$

■

### B.4.3 Normal and Geodesic Curvatures

If  $\gamma$  is a unit-speed curve on an oriented surface, then  $\dot{\gamma}$  is also a unit vector and a tangent vector of the surface. Therefore,  $\dot{\gamma}$  is perpendicular to the unit normal  $\vec{N}$  of the surface. Then  $\dot{\gamma}$ ,  $\vec{N}$  and  $\vec{N} \times \dot{\gamma}$  are mutually orthogonal unit vector. Now we can obtain

$$\ddot{\gamma} \equiv k_n \vec{N} + k_g \vec{N} \times \dot{\gamma} \quad (130)$$

because  $\ddot{\gamma}$  is perpendicular to  $\dot{\gamma}$ .

**Definition 17.** The scalars  $k_n$  is called normal curvature and  $k_g$  is called geodesic curvature of a unit-speed curve.

**Proposition 7.**

$$k_n = \ddot{\gamma} \cdot \vec{N}, \quad k_g = \ddot{\gamma} \cdot (\vec{N} \times \dot{\gamma}), \quad k^2 = k_n^2 + k_g^2, \quad k_n = k \cos \psi, \quad k_g = \pm k \sin \psi, \quad (131)$$

where  $k$  is the curvature of  $\gamma$  and  $\psi$  is the angle between  $\vec{N}$  and the principal normal  $\vec{n} \equiv \ddot{\gamma}/k$  of  $\gamma$ .

### B.4.4 The Gauss Equations

**Proposition 8.** If a surface patch  $\vec{\sigma}$  has the first and second fundamental forms as

$$Edu^2 + 2Fdudv + Gdv^2, \quad Ldu^2 + 2Mdudv + Ndv^2, \quad (132)$$



then we can get

$$\begin{aligned}
\vec{\sigma}_{uu} &= \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}\vec{\sigma}_u + \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)}\vec{\sigma}_v + L\vec{N}, \\
\vec{\sigma}_{uv} &= \frac{GE_v - FG_u}{2(EG - F^2)}\vec{\sigma}_u + \frac{EG_u - FE_v}{2(EG - F^2)}\vec{\sigma}_v + M\vec{N}, \\
\vec{\sigma}_{vv} &= \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}\vec{\sigma}_u + \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}\vec{\sigma}_v + N\vec{N}.
\end{aligned} \tag{133}$$

*Proof.* We choose a basis  $\{\vec{\sigma}_u, \vec{\sigma}_v, \vec{N}\}$  in  $R^3$  to get

$$\vec{\sigma}_{uu} = \alpha_1\vec{\sigma}_u + \alpha_2\vec{\sigma}_v + \alpha_3\vec{N}, \quad \vec{\sigma}_{uv} = \beta_1\vec{\sigma}_u + \beta_2\vec{\sigma}_v + \beta_3\vec{N}, \quad \vec{\sigma}_{vv} = \gamma_1\vec{\sigma}_u + \gamma_2\vec{\sigma}_v + \gamma_3\vec{N}. \tag{134}$$

Then we can use

$$\vec{N} \cdot \vec{\sigma}_{uu} = \alpha_3, \quad \vec{N} \cdot \vec{\sigma}_{uv} = \beta_3, \quad \vec{N} \cdot \vec{\sigma}_{vv} = \gamma_3 \tag{135}$$

to get

$$\alpha_3 = L, \quad \beta_3 = M, \quad \gamma_3 = N. \tag{136}$$

Then we determine other coefficients by the similar way as

$$\begin{aligned}
\vec{\sigma}_u \cdot \vec{\sigma}_{uu} &= \frac{1}{2}E_u = \alpha_1E + \alpha_2F, & \vec{\sigma}_v \cdot \vec{\sigma}_{uu} &= (\vec{\sigma}_u \cdot \vec{\sigma}_v)_u - \vec{\sigma}_u \cdot \vec{\sigma}_{uv} = F_u - \frac{1}{2}E_v = \alpha_1F + \alpha_2G, \\
\Rightarrow \alpha_1 &= \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, & \alpha_2 &= \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)}, \\
\vec{\sigma}_u \cdot \vec{\sigma}_{uv} &= \frac{1}{2}E_v = \beta_1E + \beta_2F, & \vec{\sigma}_v \cdot \vec{\sigma}_{uv} &= \frac{1}{2}G_u = \beta_1F + \beta_2G, \\
\Rightarrow \beta_1 &= \frac{GE_v - FG_u}{2(EG - F^2)}, & \beta_2 &= \frac{EG_u - FE_v}{2(EG - F^2)}, \\
\vec{\sigma}_u \cdot \vec{\sigma}_{vv} &= (\vec{\sigma}_u \cdot \vec{\sigma}_v)_v - \vec{\sigma}_v \cdot \vec{\sigma}_{uv} = F_v - \frac{1}{2}G_u = \gamma_1E + \gamma_2F, & \vec{\sigma}_v \cdot \vec{\sigma}_{vv} &= \frac{1}{2}G_v = \gamma_1F + \gamma_2G, \\
\Rightarrow \gamma_1 &= \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}, & \gamma_2 &= \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}.
\end{aligned} \tag{137}$$

■

#### B.4.5 Gaussian and Mean Curvatures

**Definition 18.** Let  $W_{\vec{p},S}$  be the Weingarten map of an oriented surface  $S$  at a point  $\vec{p} \in S$ . The Gaussian curvature  $K$  and mean curvature  $H$  of the oriented surface at the point  $\vec{p}$  are defined by

$$K = \det(W_{\vec{p},S}), \quad H = \frac{1}{2} \text{Tr}(W_{\vec{p},S}). \quad (138)$$

**Proposition 9.** Let  $\vec{\sigma}$  be a surface patch of an oriented surface  $S$ . Then the matrix of the Weingarten map  $W_{\vec{p},S}$  with respect to the basis  $\{\vec{\sigma}_u, \vec{\sigma}_v\}$  of  $T_p S$  is  $F_I^{-1} F_{II}$ , where

$$F_I \equiv \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad F_{II} \equiv \begin{pmatrix} L & M \\ M & N \end{pmatrix}, \quad (139)$$

if the first and second fundamental forms are

$$Edu^2 + 2Fdudv + Gdv^2, \quad Ldu^2 + 2Mdudv + Ndv^2. \quad (140)$$

*Proof.* By definitions, the Weingarten map  $W(\vec{\sigma}_u) = -\vec{N}_u$ ,  $W(\vec{\sigma}_v) = -\vec{N}_v$  so the matrix of the Weingarten map is

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad (141)$$

where

$$-\vec{N}_u = a\vec{\sigma}_u + b\vec{\sigma}_v, \quad -\vec{N}_v = c\vec{\sigma}_u + d\vec{\sigma}_v. \quad (142)$$

By using the Lemma 1, we find

$$L = aE + bF, \quad M = cE + dF, \quad M = aF + bG, \quad N = cF + dG, \quad (143)$$

or

$$F_{II} = F_I \begin{pmatrix} a & c \\ b & d \end{pmatrix}. \quad (144)$$

Thus, we show

$$W_{\vec{p},S} = F_I^{-1} F_{II}. \quad (145)$$

■

**Corollary 2.**  $K = \frac{LN-M^2}{EF-F^2}, \quad H = \frac{LG-2MF+NE}{2(EG-F^2)}.$

*Proof.* By definition, we can get

$$K = \det(F_I^{-1}F_{II}) = \frac{LN - M^2}{EG - F^2}, \quad (146)$$

and

$$F_I^{-1}F_{II} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \frac{1}{EG - F^2} \begin{pmatrix} LG - MF & MG - NF \\ ME - LF & NE - MF \end{pmatrix}. \quad (147)$$

Thus, we get

$$H = \frac{1}{2} \text{Tr}(F_I^{-1}F_{II}) = \frac{LG - 2MF + NE}{2(EG - F^2)}. \quad (148)$$

■

By using the notations in the Proposition 9 and Corollary 2, we can show

$$\begin{aligned} \vec{N}_u \times \vec{N}_v &= (a\vec{\sigma}_u + b\vec{\sigma}_v) \times (c\vec{\sigma}_u + d\vec{\sigma}_v) = (ad - bc)\vec{\sigma}_u \times \vec{\sigma}_v = \det(F_I^{-1}F_{II})\vec{\sigma}_u \times \vec{\sigma}_v \\ &= \frac{LN - M^2}{EG - F^2}\vec{\sigma}_u \times \vec{\sigma}_v = K\vec{\sigma}_u \times \vec{\sigma}_v. \end{aligned} \quad (149)$$

## B.5 The Gauss-Bonnet Theorem

We show the Gauss-Bonnet theorem for simple closed curves, curvilinear polygons and compact surfaces.

### B.5.1 The Gauss-Bonnet Theorem for Simple Closed Curves

**Definition 19.** A parametrized curve  $\gamma(t) = \vec{\sigma}(u(t), v(t)) \equiv \vec{\sigma}(\pi(t))$  on a surface patch  $\vec{\sigma} : U \rightarrow R^3$  is called a simple closed curve with period  $T$  if  $\pi(t)$  is a simple closed curve in  $R^2$  with period  $T$  such that the interior region of  $\pi$   $\text{int}(\pi)$  of  $R^2$  enclosed by  $\pi$  is entirely contained in  $U$ . The curve  $\gamma$  is positively-oriented if  $\pi$  is positively-oriented. Finally, the image of the interior region  $\text{int}(\pi)$  under the map  $\vec{\sigma}$  is defined to be the interior of  $\gamma$   $\text{int}(\gamma)$ .

**Theorem 3.** Let  $\gamma(s)$  be a unit-speed simple closed curve in a surface patch  $\vec{\sigma}$  of length  $l(\gamma)$ , and assume  $\gamma$  is positively-oriented. Then we get

$$\int_0^{l(\gamma)} ds \, k_g = 2\pi - \int_{\text{int}(\gamma)} dA_{\vec{\sigma}} K, \quad (150)$$

where  $k_g$  is the geodesic curvature of the surface  $\gamma$ ,  $K$  is the Gaussian curvature of the surface  $\vec{\sigma}$  and  $dA_{\vec{\sigma}}$  is the area element of the surface  $\vec{\sigma}$ .

*Proof.* We first consider a unit orthogonal basis  $\{e'(u, v), e''(u, v)\}$  of the tangent plane at each point of the surface patch  $\vec{\sigma}$ . Then we can find a unit orthogonal basis  $\{e', e'', \vec{N} \equiv e' \times e''\}$  in  $R^3$ . Now we assume  $\theta$  is the angle between a unit tangent vector  $\dot{\gamma}$  of the simple closed curve  $\gamma$  and the unit vector  $e'$ . Thus, we have

$$\dot{\gamma} = \cos(\theta)e' + \sin(\theta)e'', \quad \vec{N} \times \dot{\gamma} = -\sin(\theta)e' + \cos(\theta)e''. \quad (151)$$

Then we get

$$\ddot{\gamma} = \cos(\theta)\dot{e}' + \sin(\theta)\dot{e}'' + \dot{\theta}(-\sin(\theta)e' + \cos(\theta)e''). \quad (152)$$

The geodesic curvature is computed as

$$\begin{aligned} k_g &= (\vec{N} \times \dot{\gamma}) \cdot \ddot{\gamma} \\ &= \dot{\theta}(-\sin(\theta)e' + \cos(\theta)e'') \cdot (-\sin(\theta)e' + \cos(\theta)e'') \\ &\quad + (-\sin(\theta)e' + \cos(\theta)e'') \cdot (\cos(\theta)\dot{e}' + \sin(\theta)\dot{e}'') \\ &= \dot{\theta} + \cos^2(\theta)(\dot{e}' \cdot e'') - \sin^2(\theta)(\dot{e}'' \cdot e') = \dot{\theta} - e' \cdot \dot{e}''. \end{aligned} \quad (153)$$

Thus, we can get

$$\int_0^{l(\gamma)} ds \, k_g = 2\pi - \int_0^{l(\gamma)} ds \, e' \cdot \dot{e}''. \quad (154)$$

Then we can compute

$$\begin{aligned} \int_0^{l(\gamma)} ds \, e' \cdot \dot{e}'' &= \int_0^{l(\gamma)} ds \, e' \cdot (\dot{u}e''_u + \dot{v}e''_v) = \int (e' \cdot e''_u)du + (e' \cdot e''_v)dv \\ &= \int dudv \left( (e' \cdot e''_v)_u - (e' \cdot e''_u)_v \right) = \int dudv \left( (e'_u \cdot e''_v) - (e'_v \cdot e''_u) \right). \end{aligned} \quad (155)$$

To compute the final term, we use

$$e'_u = \alpha e'' + \lambda' \vec{N}, \quad e'_v = \beta e'' + \mu' \vec{N}, \quad e''_u = -\alpha' e' + \lambda'' \vec{N}, \quad e''_v = -\beta' e' + \mu'' \vec{N}, \quad (156)$$

where  $\alpha, \lambda', \beta, \mu', \alpha', \lambda'', \beta'$  and  $\mu''$  are constants. We also know

$$e'_u \cdot e'' = -e' \cdot e''_u, \quad e'_v \cdot e'' = -e' \cdot e''_v, \quad (157)$$

then we get

$$e'_u = \alpha e'' + \lambda' \vec{N}, \quad e'_v = \beta e'' + \mu' \vec{N}, \quad e''_u = -\alpha e' + \lambda'' \vec{N}, \quad e''_v = -\beta e' + \mu'' \vec{N}, \quad (158)$$

and

$$e'_u \cdot e''_v - e''_u \cdot e'_v = \lambda' \mu'' - \lambda'' \mu'. \quad (159)$$

Now we use the formula

$$\vec{N}_u \times \vec{N}_v = K \vec{\sigma}_u \times \vec{\sigma}_v, \quad (160)$$

and

$$||\vec{\sigma}_u \times \vec{\sigma}_v|| = (EG - F^2)^{\frac{1}{2}} \quad (161)$$

by the Proposition 5. Hence, we obtain

$$\vec{N}_u \times \vec{N}_v = \frac{LN - M^2}{(EG - F^2)^{\frac{1}{2}}} \vec{N} \Rightarrow (\vec{N}_u \times \vec{N}_v) \cdot \vec{N} = \frac{LN - M^2}{(EG - F^2)^{\frac{1}{2}}} \quad (162)$$

if the first and second fundamental forms are

$$Edu^2 + 2Fdudv + Gdv^2, \quad Ldu^2 + 2Mdudv + Ndv^2. \quad (163)$$

Now we compute

$$\begin{aligned} (\vec{N}_u \times \vec{N}_v) \cdot \vec{N} &= (\vec{N}_u \times \vec{N}_v) \cdot (e' \times e'') = (\vec{N}_u \cdot e')(\vec{N}_v \cdot e'') - (\vec{N}_u \cdot e'')(\vec{N}_v \cdot e') \\ &= (\vec{N} \cdot e'_u)(\vec{N} \cdot e''_v) - (\vec{N} \cdot e''_u)(\vec{N} \cdot e'_v) = \lambda' \mu'' - \lambda'' \mu'. \end{aligned} \quad (164)$$

Now we can obtain

$$e'_u \cdot e''_v - e''_u \cdot e'_v = \frac{LN - M^2}{(EG - F^2)^{\frac{1}{2}}}. \quad (165)$$

Thus, we can know

$$\begin{aligned} \int_0^{l(\gamma)} ds \, e' \cdot e'' &= \int dudv \left( (e'_u \cdot e''_v) - (e'_v \cdot e''_u) \right) = \int dudv \frac{LN - M^2}{(EG - F^2)^{\frac{1}{2}}} \\ &= \int dudv \frac{LN - M^2}{EG - F^2} (EG - F^2)^{\frac{1}{2}} = \int dA_\sigma \, K. \end{aligned} \quad (166)$$

Now we show

$$\int_0^{l(\gamma)} ds \, k_g = 2\pi - \int_{\text{int}(\gamma)} dA_\sigma \, K. \quad (167)$$

■

### B.5.2 The Gauss-Bonnet Theorem for Curvilinear Polygons

**Definition 20.** A curvilinear polygon in  $R^2$  is a continuous map  $\pi : R \rightarrow R^2$ . For real numbers  $T$ , and some points  $0 = t_0 < t_1 < t_2 < \dots < t_n = T$ :

1.  $\pi(t) = \pi(t')$  if and only if  $t' - t$  is an integer of multiple of  $T$ .
2.  $\pi$  is smooth on each of open intervals  $(t_0, t_1), (t_1, t_2), \dots, (t_{n-1}, t_n)$ .
3. One side derivatives

$$\dot{\pi}^-(t_i) \equiv \lim_{t \rightarrow t_i^-} \frac{\pi(t) - \pi(t_i)}{t - t_i}, \quad \dot{\pi}^+(t_i) = \lim_{t \rightarrow t_i^+} \frac{\pi(t) - \pi(t_i)}{t - t_i} \quad (168)$$

exist for  $i = 1, 2, \dots, n$ , and one side derivatives are non-zero and not parallel. The points  $\gamma(t_i)$  for  $i = 1, 2, \dots, n$  are called vertices of the curvilinear polygon  $\pi$ , and the segments of it corresponding to the open intervals  $(t_{i-1}, t_i)$  are called the edges of the curvilinear polygon  $\pi$ .

**Theorem 4.** Let  $\gamma$  be a positively-oriented unit-speed curvilinear polygon with  $n$  edges on a surface  $S$ , and let  $\alpha_1, \alpha_2, \dots, \alpha_n$  be the interior angle at its vertices. Then

$$\int_0^{l(\gamma)} k_g ds = \left( \sum_{i=1}^n \alpha_i \right) - (n-2)\pi - \int_{\text{int}(\gamma)} K dA_S. \quad (169)$$

*Proof.* By using the Theorem 3, we have

$$\int_0^{l(\gamma)} ds k_g = \int_0^{l(\gamma)} ds \dot{\theta} - \int_{\text{int}(\gamma)} dA_\sigma K, \quad (170)$$

where  $\theta$  is the angle between the unit tangent vector  $\dot{\gamma}$  of the curvilinear polygon  $\gamma$  and one unit vector of the curvilinear polygon  $\gamma$ . If we consider a smooth curve whose vertex has continuous derivative, then we get

$$\int_0^{l(\tilde{\gamma})} ds \dot{\tilde{\theta}} = 2\pi. \quad (171)$$

Indeed,  $\gamma$  and  $\tilde{\gamma}$  are the same except near the vertices of  $\gamma$ . Thus, we get

$$\int_0^{l(\tilde{\gamma})} ds \dot{\tilde{\theta}} - \int_0^{l(\gamma)} ds \dot{\theta} = \sum_{i=1}^n \left( \int_{s'_i}^{s''_i} ds \dot{\tilde{\theta}} - \int_{s'_i}^{s_i} ds \dot{\theta} - \int_{s_i}^{s''_i} ds \dot{\theta} \right) = \sum_{i=1}^n \delta_i, \quad (172)$$

where  $\delta_i$  is the exterior angle of the  $i$ -th vertex. We let  $s'_i$  and  $s''_i$  tend to  $s_i$  in the last equality, then we can find

$$\int_{s'_i}^{s_i} ds \dot{\theta} \rightarrow 0, \quad \int_{s_i}^{s''_i} ds \dot{\theta} \rightarrow 0, \quad \int_{s'_i}^{s''_i} ds \dot{\tilde{\theta}} \rightarrow \delta_i. \quad (173)$$

Therefore, we get

$$\begin{aligned} \int_0^{l(\gamma)} ds k_g &= 2\pi - \sum_{i=1}^n \delta_i - \int_{\text{int}(\gamma)} dA_\sigma K = 2\pi - \sum_{i=1}^n (\pi - \alpha_i) - \int_{\text{int}(\gamma)} dA_\sigma K \\ &= \left( \sum_{i=1}^n \alpha_i \right) - (n-2)\pi - \int_{\text{int}(\gamma)} dA_\sigma K. \end{aligned} \quad (174)$$

■

### B.5.3 The Gauss-Bonnet Theorem for Compact Surfaces

**Definition 21.** If we have a surface with atlas consisting of the patches  $\sigma_i : U_i \rightarrow R^3$ , then a triangulation of the surface is a collection of curvilinear polygons, each of which is contained, together with its interior, in one of  $\sigma_i$ , such that:

1. Each point of the surface is in one of the curvilinear polygons.
2. Two curvilinear polygons are either disjoint, or their intersection is a common edge or vertex.
3. Each edge is an edge of two polygons.

The compact surface has a nice property in triangulation to find Gauss-Bonnet theorem as

**Theorem 5.** *Each compact surface has a triangulation with finitely many polygons.*

**Definition 22.** *The Euler number  $\chi$  of a triangulation of a compact surface with finitely polygons is*

$$\chi = V - E + F, \quad (175)$$

where  $V$  is the total number of vertices of the triangulation,  $E$  is the total number of edges of the triangulation, and  $F$  is the total number of polygons of the triangulation.

Then we show the Gauss-Bonnet theorem for compact surface.

**Theorem 6.** *If we assume  $S$  is a compact surface, then we get*

$$\int_S K dA = 2\pi\chi, \quad (176)$$

where  $\chi$  is the Euler number of the triangulation for any triangulation of the compact surface  $S$ .

*Proof.* We first fix a triangulation of the surface  $S$  with positively-oriented unit speed curvilinear polygons  $P_i$ . Each of  $P_i$  is contained in the image of some  $\sigma_i : U_i \rightarrow R^3$  in the atlas of the surface. In other words, we have  $P_i = \sigma_i(V_i)$ , where  $V_i \in U_i$ . By the Theorem 4, we obtain

$$\int_{V_i} K dA_{\sigma_i} = s_i - (n_i - 2)\pi - \int_0^{l(\gamma_i)} k_g ds, \quad (177)$$

where  $n_i$  is the number of vertices of  $P_i$ ,  $\gamma_i$  is the positively-oriented unit-speed curvilinear polygon that forms the boundary of  $P_i$ ,  $l(\gamma_i)$  is its length, and  $s_i$  is the sum of the interior angle in  $P_i$ . Now we sum over all the curvilinear polygons  $P_i$ . We first obtain

$$\sum_i s_i = 2\pi V, \quad (178)$$

where,  $V$  is the total number of vertices, because each vertex gives  $2\pi$  angle. Then we get

$$\sum_i (n_i - 2)\pi = \left( \sum_i n_i \right) \pi - 2\pi F = 2\pi E - 2\pi F, \quad (179)$$



where  $E$  is the total number of edges and  $F$  is the total number of curvilinear polygons, because each edge in the curvilinear polygon  $P_i$  is counted twice. By the Proposition 7, the geodesic curvature  $k_g = \ddot{\gamma} \cdot (\vec{N} \times \dot{\gamma})$ , where  $\vec{N}$  is the standard unit normal, of a curve  $\gamma$  changes sign when we traverse  $\gamma$  in the opposite direction. In other word, the traverse of  $\gamma$  changes the sign of  $\dot{\gamma}$  and does not change the sign of  $\ddot{\gamma}$ . Hence, the sum of the geodesic curvature integration is canceled along each edge. Then we obtain

$$\sum_i \int_0^{l(\gamma_i)} k_g ds = 0. \quad (180)$$

Finally, we get

$$\int_S K dA = \sum_i \int_{V_i} K dA_{\sigma_i} = 2\pi(V - E + F) = 2\pi\chi. \quad (181)$$

■

We also have a mathematical theorem to show the Euler number of a compact surface is related to genus  $g$  as

**Theorem 7.** *The Euler number of a compact surface  $T_g$  is  $2 - 2g$ .*

## B.6 The Gauss and Codazzi-Mainardi Equations

**Proposition 10.** *We assume the first fundamental and second fundamental forms of a surface patch  $\sigma$  is*

$$L_v - M_u = L\beta_1 + M(\beta_2 - \alpha_1) - N\alpha_2, \quad M_v - N_u = L\gamma_1 + M(\gamma_2 - \beta_1) - N\beta_2, \quad (182)$$

where

$$\begin{aligned} \alpha_1 &= \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, & \alpha_2 &= \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)}, \\ \beta_1 &= \frac{GE_v - FG_u}{2(EG - F^2)}, & \beta_2 &= \frac{EG_u - FE_v}{2(EG - F^2)}, \\ \gamma_1 &= \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}, & \gamma_2 &= \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}. \end{aligned} \quad (183)$$

*Proof.* By using the Proposition 8, we get

$$\begin{aligned}\vec{\sigma}_{uu} &= \alpha_1 \vec{\sigma}_u + \alpha_2 \vec{\sigma}_v + L \vec{N}, \\ \vec{\sigma}_{uv} &= \beta_1 \vec{\sigma}_u + \beta_2 \vec{\sigma}_v + M \vec{N}, \\ \vec{\sigma}_{vv} &= \gamma_1 \vec{\sigma}_u + \gamma_2 \vec{\sigma}_v + N \vec{N}.\end{aligned}\tag{184}$$

Then we use  $(\vec{\sigma}_{uu})_v = (\vec{\sigma}_{uv})_u$  to get

$$\left( \alpha_1 \vec{\sigma}_u + \alpha_2 \vec{\sigma}_v + L \vec{N} \right)_v = \left( \beta_1 \vec{\sigma}_u + \beta_2 \vec{\sigma}_v + M \vec{N} \right)_u.\tag{185}$$

Then we also derive

$$\begin{aligned}& ((\alpha_1)_v - (\beta_1)_u) \vec{\sigma}_u + ((\alpha_2)_v - (\beta_2)_u) \vec{\sigma}_v + (L_v - M_u) \vec{N} \\ &= \beta_1 \left( \alpha_1 \vec{\sigma}_u + \alpha_2 \vec{\sigma}_v + L \vec{N} \right) + (\beta_2 - \alpha_1) \left( \beta_1 \vec{\sigma}_u + \beta_2 \vec{\sigma}_v + M \vec{N} \right) - \alpha_2 \left( \gamma_1 \vec{\sigma}_u + \gamma_2 \vec{\sigma}_v + N \vec{N} \right) \\ &\quad - L \vec{N}_v + M \vec{N}_u.\end{aligned}\tag{186}$$

Because  $\vec{N}$  is orthogonal to  $\vec{\sigma}_u$ ,  $\vec{\sigma}_v$ ,  $\vec{N}_u$ , and  $\vec{N}_v$ , we can use equating the coefficients of  $\vec{N}$  to get

$$L_u - M_u = L\beta_1 + M(\beta_2 - \beta_1) - N\alpha_2.\tag{187}$$

To obtain the other relation, we consider  $(\vec{\sigma}_{uv})_v = (\vec{\sigma}_{vv})_u$  to get

$$\left( \beta_1 \vec{\sigma}_u + \beta_2 \vec{\sigma}_v + M \vec{N} \right)_v = \left( \gamma_1 \vec{\sigma}_u + \gamma_2 \vec{\sigma}_v + N \vec{N} \right)_u.\tag{188}$$

Now we can derive the other relation via

$$\begin{aligned}& ((\beta_1)_v - (\gamma_1)_u) \vec{\sigma}_u + ((\beta_2)_v - (\gamma_2)_u) \vec{\sigma}_v + (M_v - N_u) \vec{N} \\ &= \gamma_1 \left( \alpha_1 \vec{\sigma}_u + \alpha_2 \vec{\sigma}_v + L \vec{N} \right) + (\gamma_2 - \beta_1) \left( \beta_1 \vec{\sigma}_u + \beta_2 \vec{\sigma}_v + M \vec{N} \right) - \beta_2 \left( \gamma_1 \vec{\sigma}_u + \gamma_2 \vec{\sigma}_v + N \vec{N} \right) \\ &\quad - M \vec{N}_v + N \vec{N}_u.\end{aligned}\tag{189}$$

By equating the coefficient of  $\vec{N}$ , we obtain

$$M_v - N_u = L\gamma_1 + M(\gamma_2 - \beta_1) - N\beta_2.\tag{190}$$

■

**Proposition 11.** *If  $K$  is the Gaussian curvature of a surface patch  $\vec{\sigma}$  with the first and second fundamental forms*

$$Edu^2 + 2Fdudv + Gdv^2, \quad Ldu^2 + 2Mdudv + Ndv^2, \quad (191)$$

*then we get*

$$\begin{aligned} EK &= (\alpha_2)_v - (\beta_2)_u + \alpha_1\beta_2 + \alpha_2\gamma_2 - \beta_1\alpha_2 - (\beta_2)^2 \\ FK &= (\beta_1)_u - (\alpha_1)_v + \beta_2\beta_1 - \alpha_2\gamma_1 \\ FK &= (\beta_2)_v - (\gamma_2)_u + \beta_1\beta_2 - \gamma_1\alpha_2 \\ GK &= (\gamma_1)_u - (\beta_1)_v + \gamma_1\alpha_1 + \gamma_2\beta_1 - (\beta_1)^2 - \beta_2\gamma_1, \end{aligned} \quad (192)$$

*where*

$$\begin{aligned} \alpha_1 &= \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, & \alpha_2 &= \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)}, \\ \beta_1 &= \frac{GE_v - FG_u}{2(EG - F^2)}, & \beta_2 &= \frac{EG_u - FE_v}{2(EG - F^2)}, \\ \gamma_1 &= \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}, & \gamma_2 &= \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}. \end{aligned} \quad (193)$$

*Proof.* We first define

$$-\vec{N}_u = a\vec{\sigma}_u + b\vec{\sigma}_v, \quad -\vec{N}_v = c\vec{\sigma}_u + d\vec{\sigma}_v, \quad (194)$$

where

$$a = \frac{LG - MF}{EG - F^2}, \quad b = \frac{ME - LF}{EG - F^2}, \quad c = \frac{MG - NF}{EG - F^2}, \quad d = \frac{NE - MF}{EG - F^2}, \quad (195)$$

from the Lemma 1. Then we use

$$\begin{aligned} & ((\alpha_1)_v - (\beta_1)_u)\vec{\sigma}_u + ((\alpha_2)_v - (\beta_2)_u)\vec{\sigma}_v + (L_v - M_u)\vec{N} \\ &= \beta_1 \left( \alpha_1\vec{\sigma}_u + \alpha_2\vec{\sigma}_v + L\vec{N} \right) + (\beta_2 - \alpha_1) \left( \beta_1\vec{\sigma}_u + \beta_2\vec{\sigma}_v + M\vec{N} \right) - \alpha_2 \left( \gamma_1\vec{\sigma}_u + \gamma_2\vec{\sigma}_v + N\vec{N} \right) \\ & \quad - L\vec{N}_v + M\vec{N}_u, \end{aligned} \quad (196)$$

and choose the coefficients of  $\vec{\sigma}_u$  to get

$$((\alpha_1)_v - (\beta_1)_u) = \beta_1\alpha_1 + (\beta_2 - \alpha_1)\beta_1 - \alpha_2\gamma_1 + Lc - Ma = \beta_2\beta_1 - \alpha_2\gamma_1 + Lc - Ma. \quad (197)$$

We also have

$$Lc - Ma = \frac{L(MG - NF) - M(LG - MF)}{EG - F^2} = -\frac{F(LN - M^2)}{EG - F^2} = -FK. \quad (198)$$

Then we can get

$$FK = (\beta_1)_u - (\alpha_1)_v + \beta_2\beta_1 - \alpha_2\gamma_1. \quad (199)$$

Now we consider the coefficients of  $\vec{\sigma}_v$  to get

$$((\alpha_2)_v - (\beta_2)_u) = \beta_1\alpha_2 + (\beta_2 - \alpha_1)\beta_2 - \alpha_2\gamma_2 + Ld - Mb. \quad (200)$$

We also compute

$$Ld - Mb = \frac{L(NE - MF) - M(ME - LF)}{EG - F^2} = \frac{E(LN - M^2)}{EG - F^2} = EK. \quad (201)$$

Then we show

$$EK = (\alpha_2)_v - (\beta_2)_u + \alpha_1\beta_2 + \alpha_2\gamma_2 - \beta_1\alpha_2 - (\beta_2)^2. \quad (202)$$

Now we use the other relation

$$\begin{aligned} & ((\beta_1)_u - (\gamma_1)_u)\vec{\sigma}_u + ((\beta_2)_v - (\gamma_2)_v)\vec{\sigma}_v + (M_v - N_u)\vec{N} \\ &= \gamma_1 \left( \alpha_1\vec{\sigma}_u + \alpha_2\vec{\sigma}_v + L\vec{N} \right) + (\gamma_2 - \beta_1) \left( \beta_1\vec{\sigma}_u + \beta_2\vec{\sigma}_v + M\vec{N} \right) - \beta_2 \left( \gamma_1\vec{\sigma}_u + \gamma_2\vec{\sigma}_v + N\vec{N} \right) \\ & \quad - M\vec{N}_v + N\vec{N}_u. \end{aligned} \quad (203)$$

to obtain

$$\begin{aligned} (\beta_1)_u - (\gamma_1)_u &= \gamma_1\alpha_1 + (\gamma_2 - \beta_1)\beta_1 - \beta_2\gamma_1 + Mc - Na, \\ (\beta_2)_v - (\gamma_2)_v &= \gamma_1\alpha_2 + (\gamma_2 - \beta_1)\beta_2 - \beta_2\gamma_2 + Md - Nb \end{aligned} \quad (204)$$

from the coefficients of  $\vec{\sigma}_u$  and  $\vec{\sigma}_v$ . We also have

$$\begin{aligned} Mc - Na &= \frac{M(MG - NF) - N(LG - MF)}{EG - F^2} = \frac{G(M^2 - NL)}{EG - F^2} = -GK, \\ Md - Nb &= \frac{M(NE - MF) - N(ME - LF)}{EG - F^2} = \frac{F(NL - M^2)}{EG - F^2} = FK. \end{aligned} \quad (205)$$

Finally, we also show

$$\begin{aligned} FK &= (\beta_2)_v - (\gamma_2)_u + \beta_1\beta_2 - \gamma_1\alpha_2 \\ GK &= (\gamma_1)_u - (\beta_1)_v + \gamma_1\alpha_1 + \gamma_2\beta_1 - (\beta_1)^2 - \beta_2\gamma_1. \end{aligned} \quad (206)$$

■

**Corollary 3.** *If the first fundamental and second fundamental forms of a surface patch  $\vec{\sigma}$  are*

$$Edu^2 + Gdv^2, \quad Ldu^2 + 2Mdudv + Ndv^2, \quad (207)$$

then

$$K = -\frac{1}{2\sqrt{EG}} \left[ \frac{\partial}{\partial u} \left( \frac{G_u}{\sqrt{EG}} \right) + \frac{\partial}{\partial v} \left( \frac{E_v}{\sqrt{EG}} \right) \right]. \quad (208)$$

*Proof.* By using the Proposition 8, we have

$$\begin{aligned} \vec{\sigma}_{uu} &= \frac{E_u}{2E} \vec{\sigma}_u - \frac{E_v}{2G} \vec{\sigma}_v + L\vec{N}, \\ \vec{\sigma}_{uv} &= \frac{E_v}{2E} \vec{\sigma}_u + \frac{G_u}{2G} \vec{\sigma}_v + M\vec{N}, \\ \vec{\sigma}_{vv} &= -\frac{G_u}{2E} \vec{\sigma}_u + \frac{G_v}{2G} \vec{\sigma}_v + N\vec{N}. \end{aligned} \quad (209)$$

Then we use the Proposition 11 to get

$$EK = -\left( \frac{E_v}{2G} \right)_v - \left( \frac{G_u}{2G} \right)_u + \frac{E_u G_u}{4EG} - \frac{E_v G_v}{4G^2} + \frac{E_v^2}{4EG} - \frac{G_u^2}{4G^2}. \quad (210)$$

Then we can compute  $K$  as in

$$\begin{aligned} -2K\sqrt{EG} &= \frac{E_{vv} + G_{uu}}{\sqrt{EG}} - \frac{E_v(EG_v + E_v G)}{2(EG)^{\frac{3}{2}}} - \frac{G_u(E_u G + EG_u)}{2(EG)^{\frac{3}{2}}} \\ &= \frac{E_{vv}}{\sqrt{EG}} - \frac{1}{2} \frac{E_v(EG)_v}{(EG)^{\frac{3}{2}}} + \frac{G_{uu}}{\sqrt{EG}} - \frac{1}{2} \frac{G_u(EG)_u}{(EG)^{\frac{3}{2}}} \\ &= \left( \frac{E_v}{\sqrt{EG}} \right)_v + \left( \frac{G_u}{\sqrt{EG}} \right)_u. \end{aligned} \quad (211)$$

Finally ,we show

$$K = -\frac{1}{2\sqrt{EG}} \left[ \frac{\partial}{\partial u} \left( \frac{G_u}{\sqrt{EG}} \right) + \frac{\partial}{\partial v} \left( \frac{E_v}{\sqrt{EG}} \right) \right]. \quad (212)$$

■

## B.7 Two Dimensional Einstein-Hilbert Theory

We show two dimensional Einstein-Hilbert action is

$$S_{EH} = -\frac{1}{4G}\chi \quad (213)$$

by showing

$$R = 2K. \quad (214)$$

Because we consider two dimensions, we use a metric

$$ds^2 = Edu^2 + Gdv^2 \quad (215)$$

without losing generality. Then we can get

$$\Gamma_{11}^1 = \frac{1}{2}E^{-1}E_u, \quad \Gamma_{11}^2 = -\frac{1}{2}G^{-1}E_v, \quad \Gamma_{12}^1 = \frac{1}{2}E^{-1}E_v, \quad \Gamma_{12}^2 = \frac{1}{2}G^{-1}G_u, \quad \Gamma_{22}^2 = \frac{1}{2}G^{-1}G_v, \quad (216)$$

$$R_{11} = -\frac{1}{2}\frac{\partial}{\partial v}\left(\frac{E_v}{G}\right) - \frac{1}{2}\frac{\partial}{\partial u}\left(\frac{G_u}{G}\right) + \frac{1}{4}\left(\frac{G_uE_u}{GE} - \frac{G_vE_v}{G^2} + \frac{E_v^2}{GE} - \frac{G_u^2}{G^2}\right), \quad (217)$$

$$R_{11} \equiv R_{121}^2 = \frac{1}{G}R_{2121} = \frac{1}{G}R_{1212} = \frac{E}{G}R_{212}^1 \equiv \frac{E}{G}R_{22}, \quad (218)$$

in where we already assumed

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}, \quad (219)$$

$$\begin{aligned} R &= \frac{1}{E}R_{11} + \frac{1}{G}R_{22} = \frac{1}{E}R_{11} + \frac{1}{E}R_{11} = \frac{2}{E}R_{11} \\ &= \frac{2}{E}\left(-\frac{1}{2}\frac{\partial_v^2 E}{G} + \frac{1}{4}\frac{G_vE_v}{G^2} - \frac{1}{2}\frac{\partial_u^2 G}{G} + \frac{1}{4}\frac{G_u^2}{G^2} + \frac{1}{4}\frac{G_uE_u}{GE} + \frac{1}{4}\frac{E_v^2}{GE}\right) \\ &= -\frac{\partial_v^2 E}{EG} + \frac{G_vE_v}{2EG^2} - \frac{\partial_u^2 G}{GE} + \frac{G_u^2}{2G^2E} + \frac{G_uE_u}{2GE^2} + \frac{E_v^2}{2E^2G} \\ &= -\frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)\right]. \end{aligned} \quad (220)$$

The last equality can shown as

$$\begin{aligned}
& -\frac{1}{\sqrt{EG}} \left[ \frac{\partial}{\partial u} \left( \frac{G_u}{\sqrt{EG}} \right) + \frac{\partial}{\partial v} \left( \frac{E_v}{\sqrt{EG}} \right) \right] \\
&= -\frac{1}{\sqrt{EG}} \left( \frac{G_{uu}}{\sqrt{EG}} - \frac{1}{2} \frac{G_u E_u}{(EG)^{\frac{3}{2}}} G - \frac{1}{2} \frac{G_u^2}{(EG)^{\frac{3}{2}}} E + \frac{E_{vv}}{\sqrt{EG}} - \frac{1}{2} \frac{G_v E_v}{(EG)^{\frac{3}{2}}} E - \frac{1}{2} \frac{E_v^2}{(EG)^{\frac{3}{2}}} G \right) \\
&= -\frac{E_{vv}}{EG} + \frac{G_v E_v}{2EG^2} - \frac{G_{uu}}{GE} + \frac{G_u^2}{2G^2 E} + \frac{G_u E_u}{2GE^2} + \frac{E_v^2}{2E^2 G}.
\end{aligned} \tag{221}$$

Thus, we get  $R = 2K$ .

## C Review of Two Dimensional Finite Entropy

We use translational invariance, Poincaré symmetry, causality and finite entanglement entropy to determine the form of two dimensional entanglement entropy [4]. We first use translational invariance to let the entanglement entropy only depends on translational invariant quantities. We choose the size of a system. Then we also have causality to guarantee the strong subadditivity [18]

$$S(A) + S(B) \geq S(A \cup B) + S(A \cap B), \quad S(A) + S(B) \geq S(A - B) + S(B - A), \tag{222}$$

where  $A$  is a Cauchy surface for casually closed sets, if  $A$  and  $B$  are Cauchy surface for casually closed sets  $B$ , then  $A$  and  $B$  are spatially separated. The strong subadditivity may not be guaranteed for interacting gauge theories [2, 7, 19]. The strong subadditivity also gives the subadditivity law [18] to obtain

$$S(\lambda x + (1 - \lambda)y) \geq \lambda S(x) + (1 - \lambda)S(y), \tag{223}$$

where  $S$  is entropy, and  $0 \leq \lambda \leq 1$ , from which the slope of the entropy does not increase. If we assume  $y > x$  and  $S(y) < S(x)$ , then we should find negative values of the entropy for larger size of a system. Thus, this implies the entropy must be non-decreasing. Now we use boost symmetry to determine the form of the entanglement entropy. We first choose the size of  $A$  system is  $\sqrt{r_1 r_2}$ , the size of  $B$  system is  $\sqrt{r_1 r_2}$ , the size of union of two systems is  $r_1$ , and the size of intersection of two systems is  $r_2$ , then we can obtain

$$S(r_1) + S(r_2) \leq 2S(\sqrt{r_1 r_2}) \tag{224}$$

via the strong subadditivity. Then we take  $r_2 \rightarrow 0$  and assume  $S(0) = \gamma$  to get

$$S(x) \leq \gamma. \quad (225)$$

Because  $S(x)$  is a non-decreasing function, the entropy must be a constant ( $\gamma$ ) for the one component set. Now we show the entropy for two component sets are also a constant. We have two systems  $A$ , which is a two component set, and  $B$ , which is a single component set, and one of two components set  $A$  is included in the  $B$ . Then we obtain

$$S(A) + \gamma \geq S(A') + \gamma, \quad (226)$$

where  $A' \equiv A \cup B$ . Therefore, this implies  $S(A) \geq S(A')$ . In other words, the entropy does not increase with increasing size of the individual component. Then we consider other systems. The system  $A$  is a two component set, the system  $B$  is a one component, the intersection of  $A$  and  $B$  is  $C$ , and  $A - B = D$ . Hence, we get

$$S(A) + \gamma \geq \gamma + S(C), \quad S(A) + \gamma \geq S(D) + \gamma. \quad (227)$$

Thus, we get

$$S(A) \geq S(C), \quad S(A) \geq S(D). \quad (228)$$

This means the entropy of two component set  $A$  is not smaller than all other two component sets  $C$  and  $D$ , and each individual component of  $C$  and  $D$  is included in  $A$ . Thus, we summarize the result, and find the entropy of two component sets should be a constant ( $\delta$ ) because the entanglement entropy only depends on the size of the system. Now we generalize this result to  $m$  components. We first consider the system  $A$  is  $m + 1$  components set, and system  $B$  is one component, which includes two components set of  $A$ . Then we get

$$S_{m+1} + \gamma \geq S_m + \delta, \quad (229)$$

where  $S_m$  is the entropy of the  $m$  component set. Now we consider the system  $A$  is a  $m$  components set, and system  $B$  is a two components set, and one individual component set of  $B$  is included in one component set of  $A$ , then we obtain

$$S_m + \delta \geq S_{m+1} + \gamma. \quad (230)$$



Hence, we get

$$S_{m+1} = S_m + \delta - \gamma. \quad (231)$$

Then we get the formula of entropy for  $m$  components as

$$S_m = \gamma + (m - 1)\beta, \quad (232)$$

where  $\beta \equiv \delta - \gamma$ . From the subadditivity, we can find

$$\gamma + (m_1 + m_2 - 1)\beta \leq 2\gamma + (m_1 + m_2 - 2)\beta \Rightarrow \beta \leq \gamma. \quad (233)$$

We also use non-negative entanglement entropy to get

$$0 \leq \gamma \leq \delta \leq 2\gamma, \quad 0 \leq \beta \leq \gamma. \quad (234)$$

## D Review of Two Dimensional $CP^{N-1}$ model

We review the two dimensional  $CP^{N-1}$  action from continuum action

$$S_{cp} = \beta N \int d^2x \left( \partial_\mu z_i^* \partial_\mu z_i + (z_i^* \partial_\mu z_i)(z_j^* \partial_\mu z_j) \right) = \beta N \int d^2x \left( \partial_\mu z_i^* \partial_\mu z_i - (z_i^* \partial_\mu z_i)(z_j \partial_\mu z_j^*) \right), \quad (235)$$

where  $z_i$  is a  $N$ -component field, which satisfies  $z_i^* z_i \equiv z \cdot z = 1$ ,  $\beta N \equiv 1/g$ , and  $g$  is the coupling constant. Then the lattice action can be written as

$$\begin{aligned} S_{cp} &\Rightarrow \beta N \sum_{x, \hat{\mu}} (z_{x+\hat{\mu}}^* - z_x^*) \cdot (z_{x+\hat{\mu}} - z_x) - (z_x^* \cdot (z_{x+\hat{\mu}} - z_x)) ((z_{x+\hat{\mu}}^* - z_x^*) \cdot z_x), \\ S_{lcp} &= -\beta N \sum_{x, \hat{\mu}} (z_x^* \cdot z_{x+\hat{\mu}}) (z_{x+\hat{\mu}}^* \cdot z_x). \end{aligned} \quad (236)$$

We can also introduce auxiliary field  $A_\mu$  to obtain

$$S_{cpg} = \beta N \int d^2x (\partial_\mu - iA_\mu) z_i^* (\partial_\mu + iA_\mu) z_i. \quad (237)$$

If we integrate out  $A_\mu$ , we can reproduce the  $S_{cp}$  by replacing  $A_\mu = iz^* \cdot \partial_\mu z$  as

$$\begin{aligned} S_{cpg} &= \beta N \int d^2x \left( \partial_\mu z^* \cdot \partial_\mu z + i(\partial_\mu z^* \cdot z) A_\mu - i(z^* \cdot \partial_\mu z) A_\mu + A_\mu A_\mu \right) \\ &= \beta N \int d^2x \left( \partial_\mu z_i^* \partial_\mu z_i - (z_i^* \partial_\mu z_i)(z_j \partial_\mu z_j^*) \right). \end{aligned} \quad (238)$$

The lattice action can be written as

$$S_{lcp} = -\beta N \sum_{x, \hat{\mu}} (z_{x+\hat{\mu}}^* \cdot z_x) U_\mu^*(x) + (z_x^* \cdot z_{x+\hat{\mu}}) U_\mu(x). \quad (239)$$

The lattice action can also be written by relating link variable to  $z$  as

$$\begin{aligned} & \beta N \sum_{x, \hat{\mu}} (z_{x+\hat{\mu}}^* \cdot z_x) \exp(-iA_\mu(x)) + (z_x^* \cdot z_{x+\hat{\mu}}) \exp(iA_\mu(x)) \\ = & \beta N |z_{x+\hat{\mu}}^* \cdot z_x| \sum_{x, \hat{\mu}} \frac{z_{x+\hat{\mu}}^* \cdot z_x}{|z_{x+\hat{\mu}}^* \cdot z_x|} \exp(-iA_\mu(x)) + \frac{z_x^* \cdot z_{x+\hat{\mu}}}{|z_x^* \cdot z_{x+\hat{\mu}}|} \exp(iA_\mu(x)), \\ & \beta \rightarrow \infty \Rightarrow \exp(iA_\mu(x)) = U_\mu(x) \rightarrow \frac{z_{x+\hat{\mu}}^* \cdot z_x}{|z_{x+\hat{\mu}}^* \cdot z_x|}. \end{aligned} \quad (240)$$

Thus, we can also use  $U_\mu(x) = z_{x+\hat{\mu}}^* \cdot z_x / |z_{x+\hat{\mu}}^* \cdot z_x|$  to write the lattice action because we assumed the continuum limit in weak coupling region. The continuum limit of two dimensional  $CP^{N-1}$  model in weak coupling region can be shown by studying  $\beta$ -function in the large  $N$  limit. To review the beta function, we first review Callan-Symanzik equation. We define  $G^{(n)}$  is connected  $n$ -point function,  $M$  is renormalization scale, and  $\lambda$  is renormalized coupling. Then we consider

$$M \rightarrow M + \delta M, \quad \lambda \rightarrow \lambda + \delta \lambda, \quad \phi \rightarrow (1 + \delta \eta) \phi, \quad G^{(n)} \rightarrow (1 + n \delta \eta) G^{(n)} \quad (241)$$

to get

$$dG^{(n)} = \frac{\partial G^{(n)}}{\partial M} \delta M + \frac{\partial G^{(n)}}{\partial \lambda} \delta \lambda = n \delta \eta G^{(n)}. \quad (242)$$

We also define

$$\beta' \equiv \frac{M}{\delta M} \delta \lambda, \quad \gamma \equiv -\frac{M}{\delta M} \delta \eta \quad (243)$$

to find Callan-Symanzik equation

$$\left( M \frac{\partial}{\partial M} + \beta' \frac{\partial}{\partial \lambda} + n \gamma \right) G^{(n)} = 0. \quad (244)$$

In the large  $N$  limit, the two dimensional  $CP^{N-1}$  model is

$$\begin{aligned} & \int Dz D\alpha \exp \left( - \int d^2x |\partial_\mu z|^2 - i \int d^2x \alpha (|z|^2 - \beta N) \right) \\ = & \int D\alpha \exp \left( - N \text{Tr} \ln(-\partial^2 + i\alpha) + i\beta N \int d^2x \alpha \right). \end{aligned} \quad (245)$$

Then we consider equations of motion for  $\alpha$  to get

$$\frac{N}{(2\pi)^2} \int d^2p \frac{1}{p^2 + i\alpha} = \beta N. \quad (246)$$

Because  $\beta N$  is a constant,  $\alpha$  must be a constant. We assume  $\alpha \equiv -im^2$ . Then we obtain

$$\frac{N}{2\pi} \ln \frac{\Lambda}{m} = \beta N, \quad (247)$$

where  $\Lambda$  is a cut-off. When we assume  $\mu$  is a renormalization scale and  $\beta_r$  is the renormalized  $\beta$ , we can get

$$\frac{N}{2\pi} \ln \frac{\Lambda}{\mu} = N(\beta - \beta_r). \quad (248)$$

Then we can find

$$m = \mu \exp(-2\pi\beta_r) = \mu \exp\left(-\frac{2\pi}{g_r N}\right), \quad g_r \equiv \frac{1}{N\beta_r}. \quad (249)$$

From the Callan-Symanzik equation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta' \frac{\partial}{\partial g_r}\right) m = 0 \Rightarrow m + \beta' m \frac{2\pi}{g_r^2 N} = 0, \quad (250)$$

we can get beta function  $\beta' = -(g_r^2 N)/(2\pi)$ . Thus, this theory is asymptotic freedom in the large  $N$  limit, and the continuum limit in two dimensional lattice  $CP^{N-1}$  is at weak coupling region in the large  $N$  limit.

## References

- [1] H. Casini, M. Huerta and J. A. Rosabal, “Remarks on entanglement entropy for gauge fields,” Phys. Rev. D **89**, no. 8, 085012 (2014) doi:10.1103/PhysRevD.89.085012 [arXiv:1312.1183 [hep-th]].
- [2] C. T. Ma, “Entanglement with Centers,” JHEP **1601**, 070 (2016) doi:10.1007/JHEP01(2016)070 [arXiv:1511.02671 [hep-th]].

- [3] D. Harlow, “Wormholes, Emergent Gauge Fields, and the Weak Gravity Conjecture,” JHEP **1601**, 122 (2016) doi:10.1007/JHEP01(2016)122 [arXiv:1510.07911 [hep-th]].
- [4] H. Casini, “Geometric entropy, area, and strong subadditivity,” Class. Quant. Grav. **21**, 2351 (2004) doi:10.1088/0264-9381/21/9/011 [hep-th/0312238].
- [5] T. Faulkner, “The Entanglement Renyi Entropies of Disjoint Intervals in AdS/CFT,” arXiv:1303.7221 [hep-th].
- [6] H. Casini and M. Huerta, “A Finite entanglement entropy and the c-theorem,” Phys. Lett. B **600**, 142 (2004) doi:10.1016/j.physletb.2004.08.072 [hep-th/0405111].
- [7] X. Huang and C. T. Ma, “Analysis of the Entanglement with Centers,” arXiv:1607.06750 [hep-th].
- [8] K. Ohmori and Y. Tachikawa, “Physics at the entangling surface,” J. Stat. Mech. **1504**, P04010 (2015) doi:10.1088/1742-5468/2015/04/P04010 [arXiv:1406.4167 [hep-th]].
- [9] J. W. Kim, “Explicit reconstruction of the entanglement wedge,” arXiv:1607.03605 [hep-th].
- [10] C. T. Ma, “Discussion of the Entanglement Entropy in Quantum Gravity,” arXiv:1609.03651 [hep-th].
- [11] J. Braun, J. W. Chen, J. Deng, J. E. Drut, B. Friman, C. T. Ma and Y. D. Tsai, “Imaginary polarization as a way to surmount the sign problem in *Ab Initio* calculations of spin-imbalanced Fermi gases,” Phys. Rev. Lett. **110**, 130404 (2013) doi:10.1103/PhysRevLett.110.130404 [arXiv:1209.3319 [cond-mat.stat-mech]]. J. W. Chen and D. B. Kaplan, “A Lattice theory for low-

- energy fermions at finite chemical potential,” *Phys. Rev. Lett.* **92**, 257002 (2004) doi:10.1103/PhysRevLett.92.257002 [hep-lat/0308016].
- [12] P. V. Buividovich and M. I. Polikarpov, “Numerical study of entanglement entropy in  $SU(2)$  lattice gauge theory,” *Nucl. Phys. B* **802**, 458 (2008) doi:10.1016/j.nuclphysb.2008.04.024 [arXiv:0802.4247 [hep-lat]].
- [13] W. Donnelly and A. C. Wall, “Do gauge fields really contribute negatively to black hole entropy?,” *Phys. Rev. D* **86**, 064042 (2012) doi:10.1103/PhysRevD.86.064042 [arXiv:1206.5831 [hep-th]].
- [14] J. C. Plefka and S. Samuel, “A Strong coupling analysis of the lattice  $CP^{N-1}$  models in the presence of a theta term,” *Phys. Rev. D* **55**, 3966 (1997) doi:10.1103/PhysRevD.55.3966 [hep-lat/9612004].
- [15] F. Bruckmann, F. Gruber, K. Jansen, M. Marinkovic, C. Urbach and M. Wagner, “Comparing topological charge definitions using topology fixing actions,” *Eur. Phys. J. A* **43**, 303 (2010) doi:10.1140/epja/i2010-10915-1 [arXiv:0905.2849 [hep-lat]].
- [16] A. Gromov and R. A. Santos, “Entanglement Entropy in 2D Non-abelian Pure Gauge Theory,” *Phys. Lett. B* **737**, 60 (2014) doi:10.1016/j.physletb.2014.08.023 [arXiv:1403.5035 [hep-th]].
- [17] H. Casini, M. Huerta and R. C. Myers, “Towards a derivation of holographic entanglement entropy,” *JHEP* **1105**, 036 (2011) doi:10.1007/JHEP05(2011)036 [arXiv:1102.0440 [hep-th]].
- [18] H. Araki and E. H. Lieb, “Entropy inequalities,” *Commun. Math. Phys.* **18**, 160 (1970). doi:10.1007/BF01646092 E. H. Lieb and M. B. Ruskai, “Proof of the strong subadditivity of quantum-mechanical entropy,” *J. Math. Phys.* **14**, 1938 (1973). doi:10.1063/1.1666274

- [19] H. Casini and M. Huerta, “Entanglement entropy for a Maxwell field: Numerical calculation on a two dimensional lattice,” *Phys. Rev. D* **90**, no. 10, 105013 (2014) doi:10.1103/PhysRevD.90.105013 [arXiv:1406.2991 [hep-th]]. K. Van Acoleyen, N. Bultinck, J. Haegeman, M. Marien, V. B. Scholz and F. Verstraete, “The entanglement of distillation for gauge theories,” arXiv:1511.04369 [quant-ph].
- [20] H. Casini and M. Huerta, “Entanglement entropy for the n-sphere,” *Phys. Lett. B* **694**, 167 (2011) doi:10.1016/j.physletb.2010.09.054 [arXiv:1007.1813 [hep-th]].
- [21] W. Fu and S. Sachdev, “Numerical study of fermion and boson models with infinite-range random interactions,” *Phys. Rev. B* **94**, no. 3, 035135 (2016) doi:10.1103/PhysRevB.94.035135 [arXiv:1603.05246 [cond-mat.str-el]].
- [22] A. Naseh, “Scale vs Conformal invariance from Entanglement Entropy,” arXiv:1607.07899 [hep-th].
- [23] D. Allahbakhshi, M. Alishahiha and A. Naseh, “Entanglement Thermodynamics,” *JHEP* **1308**, 102 (2013) doi:10.1007/JHEP08(2013)102 [arXiv:1305.2728 [hep-th]].